

# Reduction of Many-valued into Two-valued Modal Logics

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**Abstract.** In this paper we develop a 2-valued reduction of many-valued logics, into 2-valued multi-modal logics. Such an approach is based on the contextualization of many-valued logics with the introduction of higher-order Herbrand interpretation types, where we explicitly introduce the coexistence of a set of algebraic truth values of original many-valued logic, transformed as parameters (or possible worlds), and the set of classic two logic values. This approach is close to the approach used in annotated logics, but offers the possibility of using the standard semantics based on Herbrand interpretations. Moreover, it uses the properties of the higher-order Herbrand types, as their fundamental nature is based on autoreferential Kripke semantics where the possible worlds are algebraic truth-values of original many-valued logic. This autoreferential Kripke semantics, which has the possibility of flattening higher-order Herbrand interpretations into ordinary 2-valued Herbrand interpretations, gives us a clearer insight into the relationship between many-valued and 2-valued multi-modal logics. This methodology is applied to the class of many-valued Logic Programs, where reduction is done in a structural way, based on the logic structure (logic connectives) of original many-valued logics. Following this, we generalize the reduction to general structural many-valued logics, in an abstract way, based on Suszko's informal non-constructive idea. In all cases, by using developed 2-valued reductions we obtain a kind of non truth-valued modal meta-logics, where two-valued formulae are modal sentences obtained by application of particular modal operators to original many-valued formulae.

Keywords: many-valued logics, modal logics, Kripke-style semantics, paraconsistency

## 1 Introduction

A significant number of real-world applications in Artificial Intelligence have to deal with partial, imprecise and uncertain information, and that is the principal reason for introducing the non-classic many-valued logics, for example, fuzzy, bilattice-based and paraconsistent logics, etc..

In such cases we associate some *degree of belief* to ground atoms, which can be simple probability, probability interval, or other more complex data structures, as for example in Bayesian logic programs where for different kinds of atoms we associate also different (that is, different from probability) kinds of *measures*. The many-valued logics with

a set of such measures (that is, 'algebraic truth-values') are one of the main tools that we can use for such applications.

The reduction of many-valued logics into the standard 2-valued logic was considered by Suszko [1], where he illustrated how Lukasiewicz's 3-valued logic could be given a 2-valued, non truth-functional, semantics. The main point, according to Suszko, is to make a distinction between the *algebraic truth-values* in  $\mathcal{W}$  of many-valued logics, which were supposed to play a merely referential role, while only two *logical truth-values* in  $\mathbf{2} = \{0, 1\}$  (0 for false and 1 for true value) would really exist. It is also based on the fact that the abstract logic is based on a *consequence relation* that is bivalent: given a set of logic formulae  $S$ , a formula  $\phi$  can be inferred from  $S$  or not, that is, the answer to the question "if  $\phi$  is inferred from  $S$ " can only be 'Yes' or 'No'.

This point of view for 'logic values' is also considered correct by other authors, and it is also applied in the case of an ontological encapsulation [2] of many-valued algebraic logic programs into 2-valued logic programs. Moreover, in a 2-valued reduction, for any propositional formula  $\phi$  that has an 'algebraic truth-value'  $\alpha$ , we can consider a 2-valued meta-sentence "the truth-value of  $\phi$  is  $\alpha$ ", i.e.,  $t(\phi, \alpha)$  where  $t$  is a binary predicate for true sentences and  $\alpha \in \mathcal{W}$  an algebraic truth value. In order to avoid a second order logic with the formula  $t(\phi, \alpha)$ , we can transform it into a First Order (FO) formula  $[\alpha]\phi$  instead, with the introduction of a modal connective  $[\alpha]$  as in [3].

Suszko's thesis for the reduction of every tarskian (monotonic)  $n$ -valued logic into a 2-valued logic is based on this division of a set of logic values into a subset of designated and undesigned elements, but it is quite a non-constructive result. In fact, he does not explain how he obtained a 2-valued semantics, or how such a procedure could be effectively applied.

In the paper by D.Batens [4], the author proposes a sort of binary print of the algebraic truth-values for the 2-valued reduction, where each truth-value is to be put into one-to-one correspondence with one element of a set of conveniently long 'equivalent' sequences of 0's and 1's. This method is similar to what had been proposed by D.Scott a decade before [5]. But this method is not universally applicable and thus can not be effectively used. Some other authors argued against Suszko's thesis [6] using examples of paraconsistent logic and Malinowski's inferential many-valuedness. But recently in [7], based on Suszko's observations on complementarity of designated and undesigned elements, a method was exhibited for the effective implementation of Suszko's reduction to a subclass of finite-valued truth-functional logics, whose truth-values satisfy the particular assumption of separability, where the 'algebraic truth-values' can be individualized by means of the linguistic resources of the logic. What is important for the present work is that they show that a reduction of truth-functional many-valued logic into 2-valued logic will simply make it lose truth-functionality: in fact, our transformation will result in *modal* logics.

Consequently, the main contribution of this paper is to use a *constructive* approach to Suszko's method, and to exhibit a method for the effective implementation of 2-valued reduction *for all kinds* of many-valued logics. It avoids the necessity of dividing (in problematic way based on subjective opinions) a set of algebraic truth-values into designated and undesigned disjoint subsets in order to define the satisfaction relation (i.e., entailment), by using the valuations (model-theoretic semantics): the entailment

$S \models \phi$  means that every model (valuation) of  $S$  is a model of  $\phi$ . For example, any rule in a many-valued logic program  $A \leftarrow B_1, \dots, B_n$  is *satisfied* if, for a given valuation  $v$ , the algebraic truth-value of the head is greater than the value of the body, i.e., if  $v(A) \geq v(B_1 \wedge \dots \wedge B_n)$ . More discussion about this approach can be found in a new representation theorem for many-valued logics [8].

Consequently, in what follows we will consider a possible embedding of these many-valued logics into 2-valued logic, in order to understand a basic connection between them and the well investigated families of 2-valued sublanguages (logics) of the first order logic language. In the past, some approaches were made in this direction, as ad-hoc logics (for example, annotated logic), but without the real purpose of investigating this issue. We will consider the following two approaches for *predicate* many-valued logics (the propositional version can be considered as a special case, when all predicate symbols have a zero arity): the first one introduces *unary* modal operator for each truth value of original many-valued logic; the second approach introduces the *binary* modal operator for each binary truth-valued logic operator (conjunction, disjunction, implication) of original many-valued logic. Both of them will transform an original truth-functional many-valued logic into *non truth-functional* 2-valued modal logic, as follows:

1. In [9] it is shown that Fitting's 3-valued bilattice logic can be embedded into an Annotated Logic Programming that is computationally very complex and has a non standard (that is, Herbrand based) interpretation. In what follows we will use the syntactic annotation for many-valued logic programs, with a set of logic values in  $\mathcal{W}$ , where a rule of the form  $A : f(\beta_1, \dots, \beta_n) \leftarrow B_1 : \beta_1, \dots, B_n : \beta_n$ , asserts "the 'truth' of the atom  $A$  is at least (or is in)  $f(\beta_1, \dots, \beta_n) = \beta_1 \wedge \dots \wedge \beta_n$  (the result of the many-valued logic conjunction of logic values  $\beta_i \in \mathcal{W}$ ).

We will extend this consideration by introducing a *contextual* logic, which is a syntax variation of the annotated logic, where instead of annotated atoms  $B : \beta$  we will use a couple  $(B, \beta)$  that is a more practical set-based denotation and can have the Herbrand interpretations. It is the fundamental and first step when we try to transform a many-valued logic into positive 2-valued logic programs with classical conjunction and implication, where we will use modal atoms  $[\beta]B$ , ( $[\beta]$  denotes a universal modal operator), instead of annotated atoms. As we will see, such a contextualization of many-valued logic programs generates the higher-order Herbrand interpretations.

2. The ontological embedding [10] into the syntax of new encapsulated many-valued logic (in some sense meta-logic for a many-valued bilattice logic) is a 2-valued, and can be seen as a flattening of a many-valued logic, where an algebraic truth-value  $\beta \in \mathcal{W}$  of an original ground atom  $r(c_1, \dots, c_k)$  is deposited into the logic attribute of a new predicate  $r_F$ , obtained by an extension of the old predicate  $r$ , so that we obtain the 'flattened' 2-valued ground atom  $r_F(c_1, \dots, c_k, \beta)$ . In that case, we will obtain the positive multi-modal logic programs with binary modal operators for conjunction, disjunction and implication and unary modal operator for negation.

These two *knowledge invariant* 2-valued logic transformations of the original many-valued logic program are mutually inverse: we can consider the annotations as the contexts for the original atoms of the logic theory. Such a context sensitive application, with higher-order Herbrand models, can be transformed (that is, *flattened*) into the logic the-

ories with basic (ordinary) Herbrand interpretations, by enlarging the original predicates with new attributes that characterize the properties of the context: in this way a context also becomes a part of the language of a logic theory, that is, it becomes visible. The inverse of a flattening is a predicate compression [11]. In this paper we will implicitly consider only a compression of the logic attribute of the flattened predicates obtained during ontological encapsulation of a many-valued logic program: the obtained compressed predicates are identical to the predicates from the original many-valued logic program, but the value for their ground atoms is not a value of a basic set of algebraic truth-values in  $\mathcal{W}$  but a *function* (higher-order value type) in  $2^{\mathcal{W}}$  (the set of all functions from  $\mathcal{W}$  to  $2$ ). A contextualization of a many-valued logic is equivalent to the compression of logical variables of the flattened versions of many-valued logic programs.

Both approaches above are different from somewhat similar procedures investigated by Pavelka in [12] by expansion of propositional Lukasiewicz's logic with a truth-constant  $\bar{\beta}$  for every real value  $\beta \in [0, 1]$ , and successively refined by Hájek in [13] and brought to first order predicate systems in [14,15]. In fact in the first approach above we introduce not logic constants (*nullary* logic operators), but *unary* modal operators for every truth-value, while in the second approach above we introduce only new  $k$ -ary ( $k \geq 1$ ) built-in functions obtained from a semantic reflection of many-valued Herbrand interpretations of predicate many-valued logics and we enlarge the domain of values of the original logic by the set of algebraic truth-values in  $\mathcal{W}$ .

The main *motivation* of this work is a theoretical investigation of the possibility of reducing a many-valued into a standard 2-valued logic. It is not our aim to replace the original many-valued logics, which are more intuitive and natural representations used in practice. But we would like to obtain the 2-valued reductions as a canonical form for the whole family of various many-valued logics, where we can investigate their common properties and make comparisons between them. So, the main *contribution* of this article is that we present this possible canonical reduction of any many-valued into 2-valued multi-modal logic, and the possibility of reusing the rich quantity of results discovered for modal logics. In this way we also define the upper limit of the expressive power for any possible many-valued logic.

**Remark:** In what follows we are interested in general many-valued algebras, based on a lattice  $(\mathcal{W}, \leq, \wedge, \vee)$  of truth values (where ordering  $\leq$  is interpreted as truth ordering of logic values), where the meet  $\wedge$  and join  $\vee$  operators are the algebraic counterparts of logic conjunction and disjunction respectively, and extended by other unary operators (for example, by many-valued logic negation) and binary operators (for example, by many-valued logic implication). We will denote by 0 and 1 the bottom and top elements respectively of such a lattice  $\mathcal{W}$  (if  $\mathcal{W}$  is not a bounded lattice then we will add to it these two elements). Thus we are able to reduce a bounded lattice of a many-valued logic  $\mathcal{W}$  into the classic 2-valued logic with the set of logic values in  $2 = \{0, 1\} \subset \mathcal{W}$  (where  $2$  is a complete sublattice of  $\mathcal{W}$ ), in the way that the many-valued operators defined in a bounded lattice  $\mathcal{W}$  are reduced, by this two-valued reduction, into the classical 2-valued logic operators (the conjunction, disjunction, negation and material implication). Because of that, the only restriction for many-valued negation operator  $\sim$  is that  $\sim 0 = 1$  and  $\sim 1 = 0$ , such that it is antitonic (i.e., satisfies De Morgan laws be-

tween the conjunction and disjunction). The set of many-valued logic connectives will be denoted by  $\Sigma$ . Two unrelated elements  $a, b \in \mathcal{W}$  will be denoted by  $a \bowtie b$ . In order to avoid confusion between many-valued and 2-valued conjunction and disjunctions, where necessary, for 2-valued connectives we will use  $\wedge$  and  $\vee$  symbols respectively. This paper follows the following plan:

After a short introduction for 2-valued multi-modal logics, in Section 2 we present a theory for higher-order Herbrand interpretation types (and its correspondent flattening into the ordinary Herbrand interpretations) obtained in a process of contextualization by relativizing the truth (and falsity) of a logic formulae to a given context (or "possible world"). We show that this is a pre-modal development for logics and can be used directly to define 2-valued concepts with Kripke semantics. In Section 3 we present a number of significative examples for many-valued logics, and show how they can be contextualized in order to be able to introduce the logic values of a many-valued logics as particular 'logic objects' into the language of this contextual logic. The result of this contextualization (which renders visible logic values of a many-valued logic) is that the atoms in a Herbrand base have the higher-order logic values: a contextual logic has the higher-order Herbrand interpretations. We show how these higher-order Herbrand model types can be equivalently considered as multi-modal Kripke models, where a set of possible worlds is taken from the structure of these higher-order types. In Section 4 we show how these techniques can be applied to many-valued Logic Programs, and we show that they can be equivalently transformed into 2-valued multi-modal Logic Programs. We consider two kinds of transformations: the first one by introducing the set of unary modal operators for each algebraic logic value, and the second one by introducing binary modal operators in the place of the original binary many-valued logic operators. Finally, in Section 5 we develop an abstract method for a 2-valued reduction of (general) many-valued logics, transforming Suszko's non-constructive idea into a formal method. This reduction results in a non truth-functional 2-valued modal meta-logic, where 2-valued sentences are obtained by applying specific modal operators to original many-valued logic formulae.

### 1.1 Introduction to predicate multi-modal logic

A predicate multi-modal logic, for a language with a set of predicate symbols  $r \in P$  with arity  $ar(r) \geq 0$  and a set of functional symbols  $f \in F$  with arity  $ar(f) \geq 0$ , is a standard predicate logic extended by a *finite* number of universal modal operators  $\Box_i, i \geq 1$ . In this case we do not require that these universal modal operators are normal modal (that is, monotonic and multiplicative) operators as in a standard setting for modal logics, but we do require that they have the same standard Kripke semantics. In a standard Kripke semantics each modal operator  $\Box_i$  is defined by an accessibility binary relation  $\mathcal{R}_i \subseteq \mathcal{W} \times \mathcal{W}$  in a given set of possible worlds  $\mathcal{W}$ . A more exhaustive and formal introduction to modal logics and their Kripke models can easily be found in the literature, for example in [16]. Here only a short version will be given, in order to clarify the definitions used in the next paragraphs.

In what follows we denote by  $A \Rightarrow B$ , or  $B^A$ , the set of all functions from  $A$  to  $B$ , and by  $A^n$  a n-folded cartesian product  $A \times \dots \times A$  for  $n \geq 1$ .

We define the set of terms of this predicate modal logic as follows: all variables  $x \in$

$Var$ , and constants  $d \in S$  are terms; if  $f \in F$  is a functional symbol of arity  $k = ar(f)$  and  $t_1, \dots, t_k$  are terms, then  $f(t_1, \dots, t_k)$  is a term. We denote by  $\mathcal{T}_0$  the set of all ground (without variables) terms.

An atomic formula (atom) for a predicate symbol  $r \in P$  with arity  $k = ar(r)$  is an expression  $r(t_1, \dots, t_k)$ , where  $t_i, i = 1, \dots, k$  are terms. Herbrand base  $H$  is a set of all ground atoms (atoms without variables). More complex formulae, for a predicate multi-modal logic, are obtained as a free algebra obtained from the set of all atoms and usual set of classic 2-valued binary logic connectives in  $\{\wedge, \vee, \rightarrow\}$  for conjunction, disjunction and implication respectively (negation of a formula  $\phi$ , denoted by  $\neg\phi$  is expressed by  $\phi \rightarrow 0$ , where 0 is used for an inconsistent formula (has constantly value 0 for every valuation)), and a number of unary universal modal operators  $\Box_i$ . We define  $\mathcal{N} = \{1, 2, \dots, n\}$  where  $n$  is a maximal arity of symbols in the finite set  $P \cup F$ .

**Definition 1.** We denote by  $\mathcal{M} = (\mathcal{W}, \{\mathcal{R}_i \mid 1 \leq i \leq k\}, S, V)$  a multi-modal Kripke model with finite  $k \geq 1$  modal operators with a set of possible worlds  $\mathcal{W}$ , the accessibility relations  $\mathcal{R}_i \subseteq \mathcal{W} \times \mathcal{W}$ , non empty set of individuals  $S$ , and a function  $V : \mathcal{W} \times (P \cup F) \rightarrow \bigcup_{n \in \mathcal{N}} (2 \cup S)^{S^n}$ , such that for any world  $w \in \mathcal{W}$ ,

1. For any functional letter  $f \in F$ ,  $V(w, f) : S^{ar(f)} \rightarrow S$  is a function (interpretation of  $f$  in  $w$ ).
2. For any predicate letter  $r \in P$ , the function  $V(w, r) : S^{ar(r)} \rightarrow 2$  defines the extension of  $r$  in a world  $w$ ,  $\|r\| = \{\mathbf{d} = \langle d_1, \dots, d_k \rangle \in S^k \mid k = ar(r), V(w, r)(\mathbf{d}) = 1\}$ .

For any formula  $\varphi$  we define  $\mathcal{M} \models_{w,g} \varphi$  iff  $\varphi$  is satisfied in a world  $w \in \mathcal{W}$  for a given assignment  $g : Var \rightarrow S$ . For example, a given atom  $r(x_1, \dots, x_k)$  is satisfied in  $w$  by assignment  $g$ , i.e.,  $\mathcal{M} \models_{w,g} r(x_1, \dots, x_k)$ , iff  $V(w, r)(g(x_1), \dots, g(x_k)) = 1$ .

The Kripke semantics is extended to all formulae as follows:

- $$\begin{aligned} \mathcal{M} \models_{w,g} \varphi \wedge \phi & \text{ iff } \mathcal{M} \models_{w,g} \varphi \text{ and } \mathcal{M} \models_{w,g} \phi, \\ \mathcal{M} \models_{w,g} \varphi \vee \phi & \text{ iff } \mathcal{M} \models_{w,g} \varphi \text{ or } \mathcal{M} \models_{w,g} \phi, \\ \mathcal{M} \models_{w,g} \varphi \rightarrow \phi & \text{ iff } \mathcal{M} \models_{w,g} \varphi \text{ implies } \mathcal{M} \models_{w,g} \phi, \\ \mathcal{M} \models_{w,g} \Box_i \varphi & \text{ iff } \forall w' ((w, w') \in \mathcal{R}_i \text{ implies } \mathcal{M} \models_{w',g} \varphi). \end{aligned}$$

The existential modal operator  $\Diamond_i$  is equal to  $\neg \Box_i \neg$ .

A formula  $\varphi$  is said to be *true in a model*  $\mathcal{M}$  if for each assignment function  $g$  and possible world  $w$ ,  $\mathcal{M} \models_{w,g} \varphi$ . A formula is said to be *valid* if it is true in each model. We denote by  $|\phi/g| = \{w \mid \mathcal{M} \models_{w,g'} \phi/g\}$  the set of all worlds where the ground formula  $\phi/g$  (obtained from  $\phi$  and an assignment  $g$ ) is satisfied.

## 2 Contextualization: Higher-order Herbrand interpretation types

The higher-order types of Herbrand interpretations for many-valued logic programs, where we are not able to associate a fixed logic value to a given ground atom of a Herbrand base but a function in a given functional space, often arise in practice when we have to deal with uncertain information. In such cases we associate some *degree of belief* to ground atoms, which can be simple probability, probability interval, or other more complex data structures, as for example in Bayesian logic programs where for a different kind of atoms we may associate different kinds of *measures* as well.

But we can see approximate (uncertain) information as a kind of *relativization* of truth



values for sentences as follows. Let  $H$  be a Herbrand base for a logic program that handles the uncertain information, and  $r(\mathbf{d})$  a ground atom in  $H$  that logically defines a particular fact for which we have only an approximated information about when it happened. Thus, this atom  $r(\mathbf{d})$  is no longer *absolutely* true or false, but rather its truth depends on the approximate temporal information about this fact: in some time points it can be true, in other it can be false. If we consider such a temporal approximation as a *context* for this ground fact  $r(\mathbf{d}) \in H$ , then we obtain that the truth of  $r(\mathbf{d})$  is a function from the time to the ordinary set of truth values  $\mathbf{2} = \{0, 1\}$ . Consequently, the truth values of ground atoms in this Herbrand base are the functions, that is, they have a *higher-order type* (this term is taken from the typed lambda calculus) with respect to the set  $\mathbf{2}$  of truth constants. Intuitively, the approximated information is relativized to its context, and such a context further specifies the semantics for this uncertain information.

The *contextualization* is a kind of *pre-modal* Kripke modeling: in fact, if we consider a context as a Kripke "possible world", then the relativization of the truth to particular contexts is equivalent to Kripke semantics for a modal logic where the truth (or falsity) of the formulae is relativized to possible worlds. In fact, as we will see in what follows, the higher-order Herbrand models obtained by contextualization are precursors for an introduction of 2-valued epistemic concepts, that is, for a development of (absolute) 2-valued logics, and it explains their role in a 2-valued reduction of many-valued logics. The higher-order Herbrand interpretations of logic programs produce the models where the true values for ground atoms are not truth constants but functions:

**Definition 2.** [17] HIGHER-ORDER HERBRAND INTERPRETATION TYPES:

*Let  $H$  be a Herbrand base, then, the higher-order Herbrand interpretations are defined by  $I : H \rightarrow T$ , where  $T$  is a functional space  $W_1 \Rightarrow (\dots(W_n \Rightarrow \mathbf{2})\dots)$ , denoted also as  $(\dots((2^{W_n})^{W_{n-1}})\dots)^{W_1}$ , and  $W_i$ ,  $i \in [1, n]$ ,  $n \geq 1$  are the sets of parameters (the values of given domains). In the case  $n = 1$ ,  $\mathcal{W} = W_1$ ,  $T = (\mathcal{W} \Rightarrow \mathbf{2})$ , we will denote this interpretation by  $I : H \rightarrow 2^{\mathcal{W}}$ .*

In [18] there has been developed a general method of constructing 2-valued autoepistemic language concepts for each many-valued ground atom with higher-order Herbrand interpretation given in Definition 2, for which we would like to have a correspondent 2-valued logic language concept. The number of such atomic concepts to be used in the applications is always a *finite* subset  $H_M$  of  $M$  elements of the Herbrand base  $H$ .

**Definition 3.** [18] EPISTEMIC CONCEPTS: *Let  $\overline{H_M}$  be a finite sequence of  $N$  ground atoms in  $H$ ,  $H_M$  a set of elements in  $\overline{H_M}$ , and  $i_N : H_M \hookrightarrow H$  be an inclusion mapping for this finite subset of ground atoms. We define the bijection  $i_C : H_M \simeq C_M$ , with the set of derived concepts  $C_M = \{\Box_i A \mid A = \pi_i(\overline{H_M}), 1 \leq i \leq M\}$ , where  $\pi_i$  is  $i$ -th projection, such that for any ground atom  $A = \pi_i(\overline{H_M})$ ,  $i_C(A) = \Box_i A$ .*

The idea of how to pass to the possible-world Kripke semantics for modal operators  $\Box_i$ , used above for an epistemic definition of concepts, is as follows: we define the set  $Q_i = \{\mathbf{w} \mid r(\mathbf{d}) = \pi_i(\overline{H_M}) \in H \text{ and } I(r(\mathbf{d}))(\mathbf{w}) = 1\}$ . It is easy to verify that  $Q_i$  is the set of all points  $\mathbf{w} \in \mathcal{W}$  where the ground atom  $r(\mathbf{d}) = \pi_i(\overline{H_M})$ , for a given higher-order Herbrand model, is *true*. As a consequence,

we may consider  $\mathcal{W}$  as a set of possible worlds and define this higher-order Herbrand model for  $I : H \rightarrow T$  as a *Kripke model*. It follows that a higher-order language concept  $\Box_i A$  is false if and only if there is not any possible world where the ground atom  $A = \pi_i(\overline{H_M}) \in H$  is satisfied, and true if it is satisfied exactly in the set of possible worlds that defines the *meaning* of this ground atom.

We will show, in the following definition, how to define the accessibility relations for *modal* operators, used to extend an original many-valued logic by a finite set of higher-order language concepts. For example, for any ground modal atom ("concept")  $\Box_i A$ , where  $A = \pi_i(\overline{H_M})$ , we will obtain that  $|\Box_i A| \in \{\emptyset, \mathcal{W}\}$ , i.e., it is a 2-valued modal logic formula (here  $\emptyset$  is the empty set).

**Definition 4.** KRIPKE SEMANTICS FOR EPISTEMIC CONCEPTS :

Let  $I : H \rightarrow T$  be a higher-order Herbrand interpretation type, where  $T$  denotes a functional space  $W_1 \Rightarrow (\dots(W_n \Rightarrow 2)\dots)$ , with  $\mathcal{W} = W_1 \times \dots \times W_n$ , and  $P$  is the set of predicates in a Herbrand base  $H$ . Then, for a given sequence of language concepts  $\overline{H_M}$ , a quadruple  $\mathcal{M}_I = (\mathcal{W}, \{\mathcal{R}_i \mid 1 \leq i \leq M\}, S, V)$  is a Kripke model for this interpretation  $I$ , such that:

1.  $S$  is a non empty set of constants.
2. A mapping (see Definition 1)  $V : \mathcal{W} \times P \rightarrow \bigcup_{n \in \mathcal{N}} 2^{S^n}$ , such that for any  $\mathbf{w} = (w_1, \dots, w_n) \in \mathcal{W}$ ,  $r \in P$ , and  $\mathbf{d} \in S^n$  it holds:  $V(\mathbf{w}, r)(\mathbf{d}) = I(r(\mathbf{d}))(w_1) \dots (w_n)$ , where  $S^n$  denotes the set of all  $n$ -tuples of constants, and  $2^{S^n}$  the set of all functions from the set  $S^n$  to the set 2.
3. Finite set of accessibility relations: for any  $r(\mathbf{d}) = \pi_i(\overline{H_M})$ ,  $\mathcal{R}_i = \mathcal{W} \times Q_i$  if  $Q_i \neq \emptyset$ ;  $\mathcal{W} \times \mathcal{W}$  otherwise, where  $Q_i = \{\mathbf{w} \in \mathcal{W} \mid V(\mathbf{w}, r)(\mathbf{d}) = 1\}$ . Then, for any world  $\mathbf{w} \in \mathcal{W}$  and assignment  $g$ , we define the many-valued satisfaction relation, denoted by  $\mathcal{M}_I \models_{g, \mathbf{w}}$ , as follows:
  - A1.  $\mathcal{M}_I \models_{g, \mathbf{w}} r(x_1, \dots, x_n)$  iff  $V(\mathbf{w}, r)(g(x_1), \dots, g(x_n)) = 1$ , for any atom,
  - A2.  $\mathcal{M}_I \models_{g, \mathbf{w}} \Box_i r(x_1, \dots, x_n)$  iff  $\forall \mathbf{w}' ((\mathbf{w}, \mathbf{w}') \in \mathcal{R}_i \text{ implies } \mathcal{M}_I \models_{g, \mathbf{w}'} r(x_1, \dots, x_n))$ , for any ground atom  $r(\mathbf{d}) = r(g(x_1), \dots, g(x_n)) \in \pi_i(\overline{H_M})$ .

Notice that for the introduced higher-order language concepts we have that  $\mathcal{M}_I \models_{\mathbf{w}} \Box_i r(\mathbf{d})$  iff  $\forall \mathbf{w}' ((\mathbf{w}, \mathbf{w}') \in \mathcal{R}_i \text{ implies } \mathcal{M}_I \models_{\mathbf{w}'} r(\mathbf{d}))$  iff  $\pi_2(\mathcal{R}_i) = |r(\mathbf{d})|$ . Notice that we obtained the multi-modal Kripke models with universal modal operators  $\Box_i$ , that is, we obtained a kind of 2-valued reduction for a many-valued atom  $r(\mathbf{d})$ . Obviously, this technique can only be used if the number of introduced universal modal operators is *finite*.

The encapsulated information in this Kripke frame can be rendered explicit by flattening a Kripke model of this more abstract vision of data, into an ordinary Herbrand model where the original predicates are extended by set of new attributes for the hidden information.

**Definition 5.** [17] FLATTENING: Let  $I : H \rightarrow T$  be a higher-order Herbrand interpretation, where  $T$  denotes a functional space  $W_1 \Rightarrow (\dots(W_n \Rightarrow 2)\dots)$  and  $\mathcal{W} = W_1 \times \dots \times W_n$  is a cartesian product. We define its flattening into the Herbrand interpretation  $I_F : H_F \rightarrow 2$ , where  $H_F = \{r_F(\mathbf{d}, \mathbf{w}) \mid r(\mathbf{d}) \in H \text{ and } \mathbf{w} \in \mathcal{W}\}$  is the Herbrand base of predicates  $r_F$ , obtained by an extension of original predicates  $r$  by a tuple of parameters  $\mathbf{w} = (w_1, \dots, w_n)$ , such that for any  $r_F(\mathbf{d}, \mathbf{w}) \in H_F$ , it holds that  $I_F(r_F(\mathbf{d}, \mathbf{w})) = I(r(\mathbf{d}))(w_1) \dots (w_n)$ .



By this flattening of the higher-order Herbrand models we again obtain a 2-valued logic, but with a changed Herbrand base  $H_F$ . It can be used as an alternative to the introduction of universal modal operators, especially when the number of such operators is *not finite*. Both of these two approaches to the reduction of many-valued into 2-valued logics will be used in the rest of this paper, and we will show that the resulting logic in both cases is a (non truth-functional) 2-valued modal logic.

### 3 Contextualization of many-valued logics

In this Section we will apply the general results obtained in the previous Section 2 to a more specific case of many-valued-logics. This is a case of many-valued logics with uncertain, approximated or context-dependent information.

We consider only the class of many-valued logics  $\mathcal{L}_{mv}$  based on a bounded lattice  $\mathcal{W}$  of algebraic truth values, with  $\mathbf{2} \subset \mathcal{W}$ , as explained in the introduction. Then the ordering relations and operations in a bounded lattice  $\mathcal{W}$  are propagated to the function space  $\mathcal{W}^H$ , that is, to the set of all Herbrand interpretations,  $I_{mv} : H \rightarrow \mathcal{W}$ . It is straightforward [19] that this makes the function space  $\mathcal{W}^H$  itself a bounded lattice.

**Definition 6.** Let  $\mathcal{L}_{mv}$  be a many-valued logic with a set of predicate symbols  $P$ , a Herbrand base  $H$ , and with a many-valued Herbrand interpretation  $I_{mv} : H \rightarrow \mathcal{W}$ . Then its standard unique extension to all formulae is a homomorphism  $v : \mathcal{L}_{mv}^G \rightarrow \mathcal{W}$ , also called a many-valued valuation, where  $\mathcal{L}_{mv}^G$  is the subset of all ground formulae in  $\mathcal{L}_{mv}$ . That is, for any ground formula  $X, Y \in \mathcal{L}_{mv}$  holds that  $v(\sim X) = \sim v(X)$  and  $v(X \odot Y) = v(X) \odot v(Y)$ , where  $\odot$  is any binary many-valued logic connective in  $\Sigma$ .

Let us, for example, consider the following bounded lattices:

1. Fuzzy data [13,20,21]: then  $\mathcal{W} = [0, 1]$  is the *infinite* set of real numbers from 0 to 1. For any ground atom  $r(\mathbf{d}) \in H$  the  $p = I(r(\mathbf{d}))$  represents its *plausibility*. For any two  $x, y \in \mathcal{W}$ , we have that  $x \wedge y = \min\{x, y\}$ ,  $x \vee y = \max\{x, y\}$ , and negation connective  $\sim$  is determined by  $\sim x = 1 - x$ .
2. Belief quantified data [22,23,24]: then  $\mathcal{W} = \mathcal{C}[0, 1]$  is the set of all closed subintervals over  $[0, 1]$ . For any ground atom  $r(\mathbf{d}) \in H$  the  $(L, U) = I_{mv}(r(\mathbf{d}))$  represents the lower and upper bounds for expert's *belief* in  $r(\mathbf{d})$ . For any two  $[x, y], [x_1, y_1] \in \mathcal{W}$ , we have that  $[x, y] \wedge [x_1, y_1] = [\min\{x, x_1\}, \min\{y, y_1\}]$ ,  $[x, y] \vee [x_1, y_1] = [\max\{x, x_1\}, \max\{y, y_1\}]$ . The *belief* (or truth) ordering is defined as follows:  $[x, y] \leq [x_1, y_1]$  iff  $(x \leq x_1 \text{ and } y \leq y_1)$ . We define the epistemic negation [25] of a *belief*  $[x, y]$  as the *doubt*  $\sim [x, y]$ , such that  $\sim [x, y] = [\sim y, \sim x] = [1 - y, 1 - x]$ . The bottom value of this lattice is  $0 = [0, 0]$ , while the top value is  $1 = [1, 1]$ .
3. Confidence level quantified data [26,27]: then  $\mathcal{W} = \mathcal{C}[0, 1] \times \mathcal{C}[0, 1]$ . For any ground atom  $r(\mathbf{d}) \in H$  we have  $((L_1, U_1), (L_2, U_2)) = I_{mv}(r(\mathbf{d}))$ , where  $(L_1, U_1)$  represents the lower and upper bounds for expert's *belief* in  $r(\mathbf{d})$ , while  $(L_2, U_2)$

represents the lower and upper bounds for expert's *doubt* in  $r(\mathbf{d})$ , respectively.

Let  $\alpha = ([x, y], [z, v])$ ,  $\beta = ([x_1, y_1], [z_1, v_1]) \in \mathcal{W}$ , then,

$$\alpha \wedge \beta = ([\min\{x, x_1\}, \min\{y, y_1\}], [\max\{z, z_1\}, \max\{v, v_1\}]),$$

$$\alpha \vee \beta = ([\max\{x, x_1\}, \max\{y, y_1\}], [\min\{z, z_1\}, \min\{v, v_1\}]).$$

In this lattice we are interested in the ordering  $\leq$  that increases the belief and decreases the doubt of facts, that is  $([x, y], [z, v]) \leq ([x_1, y_1], [z_1, v_1])$  iff  $[x, y] \leq [x_1, y_1]$  and  $[z_1, v_1] \leq [z, v]$ .

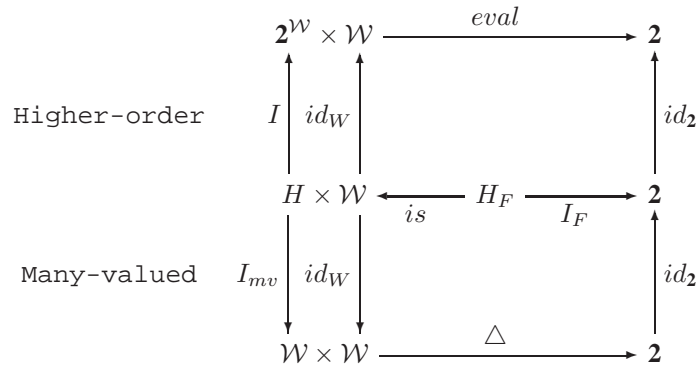
The negation  $\sim$ , which reverses this truth ordering, of this lattice is defined by Ginsberg [25], with  $\sim([x, y], [z, v]) = ([z, v], [x, y])$ . The bottom value of this lattice is  $0 = ([0, 0], [1, 1])$ , while the top value is  $1 = ([1, 1], [0, 0])$ .

4. Belnap's bilattice based logic programs [28]: Then its truth lattice is  $\mathcal{W} = \mathcal{B} = \{f, t, \top, \perp\}$ , where  $1 = t$  is *true*,  $0 = f$  is *false*,  $\top$  is inconsistent (both true and false) or *possible*, and  $\perp$  is *unknown*. As Belnap observed, these values can be given a *truth* ordering,  $\leq_t$ , such that  $0 \leq_t \top \leq_t 1$ ,  $0 \leq_t \perp \leq_t 1$  and  $\perp \bowtie_t \top$ , with  $\alpha \wedge \beta = \min_t\{\alpha, \beta\}$ ,  $\alpha \vee \beta = \max_t\{\alpha, \beta\}$ , and the epistemic negation  $\sim$  is defined by:  $\sim 0 = 1$ ,  $\sim 1 = 0$ ,  $\sim \perp = \perp$ ,  $\sim \top = \top$ .

All examples above are more than bounded lattices: they are complete distributive lattices [29,30]. Thus, we consider also that for any two elements  $a, b \in \mathcal{W}$  the many-valued implication  $a \rightarrow b$  for complete lattices can be defined as a reduct (the relative pseudocomplement), that is  $a \rightarrow b = \vee\{c \in \mathcal{W} \mid c \wedge a \leq b\}$ , so that  $a \rightarrow b = 1$  iff  $1 \wedge a = a \leq b$ .

For a given *many-valued* logic  $\mathcal{L}_{mv}$ , we can generate a *contextual* logic  $\mathcal{L}_{ct}$ , so that for any ground atom  $r(\mathbf{d}) \in H$  with a logic value  $\mathbf{w} = I_{mv}(r(\mathbf{d}))$ , we generate a *contextual atom*, a couple  $(r(\mathbf{d}), \mathbf{w}) \in H \times \mathcal{W}$ , which tell us that "the atom  $r(\mathbf{d})$  in the context  $\mathbf{w}$  is true". We also define the extended Herbrand base  $H_F = \{r_F(\mathbf{d}, \mathbf{w}) \mid r(\mathbf{d}) \in H \text{ and } \mathbf{w} \in \mathcal{W}\}$  by extending each original atom by the logic attribute with the domain  $\mathcal{W}$ , and with the bijection  $is : H_F \rightarrow H \times \mathcal{W}$ , such that for any extended (or flattened) ground atom  $r_F(\mathbf{d}, \mathbf{w}) \in H_F$  it holds that  $is(r_F(\mathbf{d}, \mathbf{w})) = (r(\mathbf{d}), \mathbf{w})$ .

This *contextualization* of a many-valued logic can be represented by the following commutative diagram



where  $eval$  is the application of the first argument (function) to the second argument,  $id$ 's are the identities, and  $\Delta$  is the 'diagonal' function, such that  $\Delta(\mathbf{w}, \mathbf{w}') = 1$

iff  $\mathbf{w} = \mathbf{w}'$ , so that the higher-order Herbrand interpretation is obtained from a many-valued Herbrand interpretation by  $I = [\Delta \circ (I_{mv} \times id_W)]$ , where  $[\_]$  is the currying ( $\lambda$  abstraction) operator for functions. The flattened Herbrand interpretation (of a 'meta' logic obtained by an ontological encapsulation of original many-valued logic), is equal to:  $I_F = eval \circ ([\Delta \circ (I_{mv} \times id_W)] \times id_W) \circ is$ .

Intuitively, the diagram above shows that for any many-valued interpretation  $I_{mv}$ , we obtain the correspondent 2-valued interpretation  $I_F$  (but with modified Herbrand base  $H_F$ ), and, equivalent to it, the higher-order Herbrand interpretation  $I$ .

By this contextualization of a many-valued logic we obtain the simplest case of the higher-order Herbrand interpretation given by Definition 2,  $I : H \rightarrow 2^{\mathcal{W}}$ , such that for any atom  $r(\mathbf{d}) \in H$  and  $\mathbf{w} \in \mathcal{W}$  holds that:

$$I(r(\mathbf{d}))(\mathbf{w}) = 1, \text{ iff } \mathbf{w} = I_{mv}(r(\mathbf{d})).$$

The accessibility relations  $\mathcal{R}_i = \mathcal{W} \times Q_i$ , for any  $r(\mathbf{d}) = \pi_i(\overline{H_M}) \in H$ , in Definition 4 for many-valued logic does not depend on the number of ground atoms in a Herbrand base, but only on the number of logic values in  $\mathcal{W}$ : it results from the fact that to any ground atom in a *consistent* many-valued logic we can assign only *one* logic value, so that  $Q_i = \{\mathbf{w} \in \mathcal{W} \mid r(\mathbf{d}) = \pi_i(\overline{H_M}) \in H \text{ and } I(r(\mathbf{d}))(\mathbf{w}) = 1\} = \{\mathbf{w}\}$  is a singleton, with  $\mathbf{w} = I_{mv}(r(\mathbf{d}))$ .

Thus, we are able to make the reduction to 2-valued logic by the introduction of a number of universal modal operators  $\Box_{\mathbf{w}}$  (denoted also by  $[\mathbf{w}]$  in what follows) with the accessibility relation  $\mathcal{R}_{\mathbf{w}} = \mathcal{W} \times Q_{\mathbf{w}} = \mathcal{W} \times \{\mathbf{w}\}$ , for each  $\mathbf{w} \in \mathcal{W}$ .

Each universal modal operator  $[\mathbf{w}]$ , with the meaning "has the value  $\mathbf{w}$ ", is defined algebraically in a lattice  $\mathcal{W}$  as a unary operator (function)  $[\mathbf{w}] : \mathcal{W} \rightarrow 2 \subseteq \mathcal{W}$ , such that for any  $\mathbf{w}_1 \in \mathcal{W}$ ,  $[\mathbf{w}](\mathbf{w}_1) = 1$  if  $\mathbf{w}_1 = \mathbf{w}$ ; 0 otherwise.

These modal operators *are not monotonic* operators, so that we obtain a non-normal Kripke modal logic (for example, the necessity rule does not hold).

As we can see, we assume that the set of possible worlds of the relational Kripke frames, used for the transformation of many-valued into multi-modal 2-valued logic, is the set of logic values of this many-valued logic. This is an autoreferential semantics [31,32] and a formal result of the modal transformation for higher-order Herbrand models and the transformation of many-valued Herbrand models into higher-order Herbrand models. The philosophical assumption is, instead, that each possible world represents a level of *credibility*, so that only the propositions with the right logic value (i.e., level of credibility) can be accepted by this world.

## 4 Reduction of many-valued into 2-valued multi-modal Logic Programs

Let  $PR$  be a many-valued logic program, for a given many-valued logic  $\mathbb{L}_{mv}$  with a set of algebraic truth-values given by a bounded lattice  $\mathcal{W}$ , a Herbrand base  $H$  and a many-valued Herbrand interpretation  $I_{mv} : H \rightarrow \mathcal{W}$  that is also a *model* of  $PR$ , i.e., an interpretation that satisfies all logic clauses in a logic program  $PR$ . We denote by  $Mod$  the subset of all Herbrand interpretations in  $\mathcal{W}^H$  that are also models of  $PR$ . Then we will have the following two cases:

1. In the first case by introducing the set of unary modal operators for each algebraic

logic value in  $\mathcal{W}$  (both for finite and infinite cases) we obtain the *standard* 2-valued modal logic for the satisfaction of logic conjunction and disjunction (if a proposition is defined by the set of worlds where it is satisfied, then the conjunction/disjunction of any two propositions is equal to the set intersection/union respectively), by transforming many-valued ground atoms into 2-valued *modal* ground atoms.

2. In the second case we do not use one specific unary modal operator for each given algebraic logic value, which can be somewhat complex issue when the cardinality of  $\mathcal{W}$  is very big or infinite. We do not transform the many-valued logic connectives into the standard 2-valued logic connectives as in the first case: instead, they will be transformed into *binary* modal operators with the ternary accessibility relations. In order to obtain a non standard modal logic in which the intersection/union properties hold for conjunction/disjunction respectively, we also need to introduce an existential modal operator with binary accessibility relation equal to the cartesian product of possible worlds. The semantics of this approach is more complex and transforms all original atoms of the many-valued logic, but offers one advantage because the number of modal operators is small, equal to the number of logic operators in the original many-valued logic.

#### 4.1 Unary modal operators case

We will show how a many-valued logic program can be transformed into the 2-valued multi-modal logic program *without* modifying the original set of atoms of a many-valued logic program.

As we have seen, by the contextualization of a many-valued logic  $\mathbb{L}_{mv}$  we obtain a contextual logic  $\mathbb{L}_{ct}$  with the same Herbrand base  $H$  as the original many-valued logic but (for a given many-valued Herbrand model  $I_{mv} \in Mod$ ) with a higher-order model  $I = [\Delta \circ (I_{mv} \times id_W)] : H \rightarrow 2^{\mathcal{W}}$  as has been shown by the commutative diagram in Section 3. We are now able to apply the result of the method in Definition 4 to this contextual logic with higher-order model types.

A simple modal formulae  $[w]p(x_1, \dots, x_n)$ , where  $w \in \mathcal{W}$  and  $p(x_1, \dots, x_n)$  is an atom of the many-valued logic program  $PR$ , will be called *m-atom* (*modal atom*). A 2-valued multi-modal logic, obtained by the substitution of original many-valued atoms by these m-atoms, is considered the first time in the case of the 4-valued Belnap's logics, used for databases with incomplete and inconsistent information [10].

**Definition 7.** (*Program Transformation: Syntax*) Let  $PR$  be a many-valued lattice-based logic program. We define its transformation in the correspondent positive multi-modal logic program  $P_{mm}$  as follows (**bold constants and variables denote tuples**):

1. Each ground atom in the original many-valued program  $PR$ ,  $p(\mathbf{c}) \leftarrow \alpha$ , where  $\alpha \in \mathcal{W}$  is a fixed logic value, we transform into the following 2-valued ground m-atom clause in  $P_{mm}$ : (1)  $[\alpha]p(\mathbf{c}) \leftarrow$

2. Each set of original many-valued clauses in  $PR$ , with the same head, (here  $\vee, \wedge$  are a many-valued disjunction and conjunction respectively, i.e., the join and meet operators of a lattice  $\mathcal{W}$ , and  $S$  is a finite interval of natural numbers from 1 to  $n$ ),

$p(\mathbf{x}) \leftarrow \bigvee_{j \in S} (r_{j,1}(\mathbf{x}_{j,1}), \dots, r_{j,k_j}(\mathbf{x}_{j,k_j}), \sim r_{j,k_j+1}(\mathbf{x}_{j,k_j+1}), \dots, \sim r_{j,m_j}(\mathbf{x}_{j,m_j}))$ , we transform as follows:

let us denote by  $Var_w = \bigcup_{j \in S} \{v_{j,1}, \dots, v_{j,k_j}, v_{j,k_j+1}, \dots, v_{j,m_j}\}$  the set of logic variables for atoms in this clause. Then, for each assignment  $g : Var_w \rightarrow \mathcal{W}$  we define

a new 2-valued clause with  $m$ -atoms, and with the classic 2-valued disjunction  $\vee$ , in  $P_{mm}$  :

$$(2) \quad [\beta]p(\mathbf{x}) \leftarrow \bigvee_{j \in S} ([\alpha_{j,1}]r_{j,1}(\mathbf{x}_{j,1}), \dots, [\alpha_{j,k_j}]r_{j,k_j}(\mathbf{x}_{j,k_j}), \\ , [\alpha_{j,k_j+1}]r_{j,k_j+1}(\mathbf{x}_{j,k_j+1}), \dots, [\alpha_{j,m_j}]r_{j,m_j}(\mathbf{x}_{j,m_j})), \\ \text{where } \alpha_{j,i} = g(v_{j,i}), \text{ for } j \in S, 1 \leq i \leq m_j, \text{ and} \\ \beta = \bigvee_{j \in S} (g(v_{j,1}) \wedge \dots \wedge g(v_{j,k_j}) \wedge \sim g(v_{j,k_j+1}) \wedge \dots \wedge \sim g(v_{j,m_j})).$$

Consequently, based on clauses (1) and (2), we obtain a standard positive logic program  $P_{mm}$  with 2-valued  $m$ -atoms.

**Remark:** notice that the obtained positive multi-modal logic program  $P_{mm}$  uses only standard 2-valued logic connectives, in contrast to the original many-valued logic program  $PR$  where the logic connectives are lattice-based (many-valued) logic operators. The grounded program  $P_{mm}^G$  obtained from the program  $P_{mm}$ , by substituting in all possible ways the variables of its atoms in all its clauses, will contain only 'modal ground atoms'  $[\alpha_{k_j}]r_{j,k_j}(\mathbf{d}_{j,k_j})$ . To such atomic formulae we can assign the new fresh propositional symbols, so that with these propositional symbols the program  $P_{mm}^G$  becomes a pure 2-valued logic program.

As we can verify, the obtained multi-modal logic program  $P_{mm}$  is a *positive* logic program (without negation in the body of clauses), so that it has a *unique* model (the set of all true facts derivable from this 2-valued logic program).

**Proposition 1 (Invariance)** *For any given many-valued logic program  $PR$ , the transformed 2-valued logic program  $P_{mm}$  with modal atoms has the same Herbrand model  $I_{mv} : H \rightarrow \mathcal{W}$  as the original program  $PR$ .*

**Proof:** We have to show that for a given logic program  $PR$ , its many-valued Herbrand model  $I_{mv} : H \rightarrow \mathcal{W}$  also satisfies the clauses of the positive 2-valued modal program  $P_{mm}$ . We will consider their grounded versions,  $PR_G$  and  $P_{mm}^G$  respectively. Then, for any ground fact  $p(\mathbf{c}) \leftarrow \alpha$  we have that  $I_{mv}(p(\mathbf{c})) = \alpha$ , so that for the modal operator  $[\alpha] : \mathcal{W} \rightarrow \mathbf{2}$ ,  $[\alpha](I_{mv}(p(\mathbf{c}))) = [\alpha](\alpha) = 1$  and, consequently, the correspondent modal fact in  $P_{mm}^G$ ,  $[\alpha]p(\mathbf{c}) \leftarrow$ , is satisfied by  $I_{mv}$ .

Let us consider a ground clause in  $PR_G$ ,

$$(1) \quad p(\mathbf{c}) \leftarrow \bigvee_{j \in S} (r_{j,1}(\mathbf{c}_{j,1}), \dots, r_{j,k_j}(\mathbf{c}_{j,k_j}), \sim r_{j,k_j+1}(\mathbf{c}_{j,k_j+1}), \dots, \sim r_{j,m_j}(\mathbf{c}_{j,m_j})),$$

which is satisfied by the model  $I_{mv}$  with logic values  $w = I_{mv}(p(\mathbf{c}))$  and  $w_{j,i_j} = I_{mv}(r_{j,i_j}(\mathbf{c}_{j,i_j}))$  for  $1 \leq j \leq n$  and  $1 \leq i \leq m$ , such that  $w = \bigvee_{j \in S} (w_{j,1} \wedge \dots \wedge w_{j,k_j} \wedge \sim w_{j,k_j+1} \wedge \dots \wedge \sim w_{j,m_j})$ . Then, the transformation of this ground clause (1) of  $PR_G$  into the 2-valued modal clauses will be the following *set* of modal rules

$$(2) \quad [\beta]p(\mathbf{c}) \leftarrow \bigvee_{j \in S} ([\alpha_{j,1}]r_{j,1}(\mathbf{c}_{j,1}), \dots, [\alpha_{j,k_j}]r_{j,k_j}(\mathbf{c}_{j,k_j}), \\ , [\alpha_{j,k_j+1}]r_{j,k_j+1}(\mathbf{c}_{j,k_j+1}), \dots, [\alpha_{j,m_j}]r_{j,m_j}(\mathbf{c}_{j,m_j})),$$

for all combinations of  $\alpha_{j,i} \in \mathcal{W}$ , for  $j \in S$ ,  $1 \leq i \leq m_j$ , and

$$\beta = \bigvee_{j \in S} (\alpha_{j,1} \wedge \dots \wedge \alpha_{j,k_j} \wedge \sim \alpha_{j,k_j+1} \wedge \dots \wedge \sim \alpha_{j,m_j}).$$

It is easy to verify that the body of the ground rule (2) is true only iff  $\alpha_{j,i_j} = w_{j,i_j}$  for all  $j \in S$  and  $1 \leq i \leq m$ , and that in that case  $\beta = w$ , so that  $[\beta](I_{mv}p(\mathbf{c})) = [\beta](w) = [\beta](\beta) = 1$ , that is, also the head of this rule is true, so that this clause is satisfied. For any other combination of modal operators in every other rule (2), derived from the rule (1), we obtain that its body is false, thus such a rule in  $P_{mm}$  is satisfied by  $I_{mv}$ . Consequently, the Herbrand model  $I_{mv}$  also satisfies the transformed 2-valued

logic program  $P_{mm}$ . From the fact that  $P_{mm}$ , as a positive logic program, can have only one Herbrand model we conclude that  $I_{mv}$  is the *unique* Herbrand model of  $P_{mm}$ , and thus the program transformation is correct and knowledge invariant.

□

Now we will consider a Kripke model for this transformed modal 2-valued logic program  $P_{mm}$  based on Definition 4, based on the particular accessibility relations for introduced unary modal operators previously discussed.

**Definition 8.** (*Program Transformation: Semantics*)

Let  $PR$  be a many-valued lattice-based logic program with a many-valued Herbrand model  $I_{mv} : H \rightarrow \mathcal{W}$ , where  $H$  is a Herbrand base with a set  $P$  of predicate symbols. Its correspondent positive multi-modal logic program  $P_{mm}$  has the Kripke model  $\mathcal{M}_I = (\mathcal{W}, \{\mathcal{R}_w \mid w \in \mathcal{W}\}, S, V)$ , with  $\mathcal{R}_w = \mathcal{W} \times \{w\}$  and  $V : \mathcal{W} \times P \rightarrow \bigcup_{n \in \mathcal{N}} 2^{S^n}$  (from Definition 1), such that for any  $p \in P$  with arity  $n$ , a set of constants  $(c_1, \dots, c_n) \in S^n$ , and a world  $w \in \mathcal{W}$ ,  $V(w, p)(c_1, \dots, c_n) = 1$  iff  $w = I_{mv}(p(c_1, \dots, c_n))$ .

Then, for any assignment  $g$  and  $w \in \mathcal{W}$ , the satisfaction relation  $\models_{g,w}$  is defined as follows:

1.  $\mathcal{M}_I \models_{g,w} p(x_1, \dots, x_n)$  iff  $V(w, p)(g(x_1), \dots, g(x_n)) = 1$ .
2.  $\mathcal{M}_I \models_{g,w} [\alpha]p(x_1, \dots, x_n)$  iff  $\forall y((w, y) \in \mathcal{R}_\alpha \text{ implies } \mathcal{M}_I \models_{g,y} p(x_1, \dots, x_n))$ .
3.  $\mathcal{M}_I \models_{g,w} \phi \wedge \psi$  iff  $\mathcal{M}_I \models_{g,w} \phi$  and  $\mathcal{M}_I \models_{g,w} \psi$ .
4.  $\mathcal{M}_I \models_{g,w} \phi \vee \psi$  iff  $\mathcal{M}_I \models_{g,w} \phi$  or  $\mathcal{M}_I \models_{g,w} \psi$ .
5.  $\mathcal{M}_I \models_{g,w} \phi \rightarrow \psi$  iff  $\mathcal{M}_I \models_{g,w} \phi$  implies  $\mathcal{M}_I \models_{g,w} \psi$ .

**Remark:** We obtained a modal logic for the multi-modal program  $P_{mm}$  in Definition 7. If we denote by  $|\psi/g|$  the set of worlds where the ground formula  $\psi/g$  is satisfied, then  $|p(g(x_1), \dots, g(x_n))|$  is a singleton set.

Thus, differently from the original ground atoms that can be satisfied in a singleton set only, the modal atoms have a standard 2-value property, that is, they are true or false in the Kripke model, and consequently are satisfiable in all possible worlds, or absolutely not satisfiable in any world. Consequently, our positive modal program with modal atoms satisfies the classic 2-valued properties:

**Proposition 2** For any ground formula  $\phi/g$  of a positive multi-modal logic program  $P_{mm}$  in Definition 7, we have that  $|\phi/g| \in \{\emptyset, \mathcal{W}\}$ , where  $\emptyset$  is the empty set.

**Proof:** by structural induction :

1.  $|\llbracket \alpha \rrbracket p(x_1, \dots, x_n)/g| = \mathcal{W}$  if  $\alpha = I_{mv}(p(g(x_1), \dots, g(x_n)))$ ;  $\emptyset$ , otherwise.
- Let, by inductive hypothesis,  $|\phi/g|, |\psi/g| \in \{\emptyset, \mathcal{W}\}$ , then
2.  $\mathcal{M}_I \models_{g,w} \phi \wedge \psi$  iff  $w \in |(\psi \wedge \phi)/g| = |\psi/g| \cap |\phi/g| \in \{\emptyset, \mathcal{W}\}$ .
  3.  $\mathcal{M}_I \models_{g,w} \phi \vee \psi$  iff  $w \in |(\psi \vee \phi)/g| = |\psi/g| \cup |\phi/g| \in \{\emptyset, \mathcal{W}\}$ .
  4.  $\mathcal{M}_I \models_{g,w} \phi \rightarrow \psi$  iff  $w \in |\phi/g|$  implies  $w \in |\psi/g|$ . Thus,  $\phi \rightarrow \psi$  is true in the model  $\mathcal{M}_I$  iff  $|\phi/g| \subseteq |\psi/g|$ , that is, iff  $v_B(\phi/g) \leq v_B(\psi/g)$ , or, alternatively,  $\phi/g \vdash \psi/g$ , where  $\vdash$  is the deductive inference relation for this 2-valued modal logic. Thus,  $|\phi/g \rightarrow \psi/g| = (\mathcal{W} - |\phi/g|) \cup |\psi/g| \in \{\emptyset, \mathcal{W}\}$ .

□

The following proposition demonstrates the existence of a one-to-one correspondence



between the unique many-valued model of the original many-valued logic program  $P$  and this unique multi-modal positive logic program  $P_{mm}$ .

**Proposition 3** *Let  $PR$  be a many-valued logic program with a Herbrand model  $I_{mv} : H \rightarrow \mathcal{W}$ , then the model  $\mathcal{M}_I$  of the 2-valued multi-modal program  $P_{mm}$ , obtained by the transformation defined in Definition 7, is composed by the set of true atomic formulae  $S_T = \{[\alpha]p(\mathbf{c}) \mid p(\mathbf{c}) \in H \text{ and } \alpha = I_{mv}(p(\mathbf{c}))\}$ .*

**Proof:** For any ground atom  $p(\mathbf{c}) \in H$  such that its logic value in a Herbrand model  $I_{mv} : H \rightarrow \mathcal{W}$ , obtained by Clark's completion [33,34,35], is equal to  $\alpha = I_{mv}(p(\mathbf{c}))$ , we have that  $|\alpha]p(\mathbf{c})| = \{w \in \mathcal{W} \mid \mathcal{M}_I \models_w [\alpha]p(\mathbf{c})\} = \{w \in \mathcal{W} \mid \alpha = I_{mv}(p(\mathbf{c}))\} = \mathcal{W}$ . Thus,  $[\alpha]p(\mathbf{c})$  is true in  $\mathcal{M}_I$ .  $\square$

**Remark:** This transformation of many-valued logic programs into *positive* (without negation) logic programs (but with *modal* atoms), can also be used to manage the *inconsistency* in 2-valued logic programs: while in the original 2-valued logic we are not able to manage the ground atom  $p(c_1, \dots, c_n)$  that is both true and false without the explosive inconsistency of all logic, in the transformed positive modal program we can have the ground modal atoms  $[1]p(c_1, \dots, c_n)$  and  $[0]p(c_1, \dots, c_n)$  both true without generating the inconsistency. This means that this kind of 2-valued transformation can be used for *paraconsistent* logics, as shown in the example below.

**Example 1:** The smallest *nontrivial* bilattice is Belnap's 4-valued bilattice [36,28]  $\mathcal{W} = \mathcal{B} = \{f, t, \perp, \top\}$  where  $t$  is *true*,  $f$  is *false*,  $\top$  is *inconsistent* (both true and false) or *possible*, and  $\perp$  is *unknown*. As Belnap observed, these values can be given two natural orders: *truth* order,  $\leq_t$ , and *knowledge* order,  $\leq_k$ , such that  $f \leq_t \top \leq_t t$ ,  $f \leq_t \perp \leq_t t$ ,  $\perp \bowtie_t \top$  and  $\perp \leq_k f \leq_k \top$ ,  $\perp \leq_k t \leq_k \top$ ,  $f \bowtie_k t$ . That is, the bottom element 0 for  $\leq_t$  ordering is  $f$ , and for  $\leq_k$  ordering is  $\perp$ , and the top element 1 for  $\leq_t$  ordering is  $t$ , and for  $\leq_k$  ordering is  $\top$ .

Meet and join operators under  $\leq_t$  are denoted  $\wedge$  and  $\vee$ ; they are natural generalizations of the usual conjunction and disjunction notions. Meet and join under  $\leq_k$  are denoted  $\otimes$  and  $\oplus$ , such that it holds that:  $f \otimes t = \perp$ ,  $f \oplus t = \top$ ,  $\top \wedge \perp = f$  and  $\top \vee \perp = t$ .

We may use a *relative pseudo-complements* for the implication, defined by  $x \rightarrow y = \vee\{z \mid z \wedge x \leq_t y\}$ , and the *pseudo-complements* for the negation,  $\neg_t x = x \rightarrow f$ .

In Belnap's bilattice the conflation  $-$  is a monotone function that preserves all finite meets (and joins) w.r.t. the lattice  $(\mathcal{B}, \leq_t)$ , thus it is the universal (and existential, because  $- = \neg_t - \neg_t$ ) modal many-valued operator: "it is believed that", which extends the 2-valued belief of the autoepistemic logic as follows:

1. if  $A$  is true than "it is believed that  $A$ ", i.e.,  $-A$ , is true;
  2. if  $A$  is false than "it is believed that  $A$ " is false;
  3. if  $A$  is unknown than "it is believed that  $A$ " is inconsistent: it is really inconsistent to believe in something that is unknown;
  4. if  $A$  is inconsistent (that is, *both* true and false) then "it is believed that  $A$ " is unknown: really, we can not tell anything about belief in something that is inconsistent.
- This belief modal operator is used to define the *epistemic negation*  $\neg$ , as composition of the strong negation  $\neg_t$  and this belief operator, i.e.,  $\neg = \neg_t -$ .

Let us show how these modal atoms in Definitions 7 and 8 can be used for paracon-

sistent logic, able to deal with the truth of the formulae  $B = A \wedge \neg A$  as well: when a formula  $B$  is *true* then a formula  $A$  is called *inconsistent*, that is, has the logic value  $\top$  in the Belnap's 4-valued logic. It is easy to see that in such a case a formula  $B$  corresponds to a 2-valued formula  $[\top]A$ , i.e.,  $[\top]A = A \wedge \neg A$ , where the modal operator  $[\top]$  is an "it is inconsistent" operator (used also as  $\bullet$  in Logics of Formal Inconsistency (LFI) [37] for the 3-valued sublattice  $\mathcal{B}_3 = \{f, \top, t\} \subset \mathcal{B}_4$ ).

But the other operator  $[\perp]$  is a modal "it is *unknown*" operator, used to support an incomplete knowledge as well. That is, when a formula  $[\perp]A$  is true, then a formula  $A$  is called *unknown*, and has the value  $\perp$  in Belnap's 4-valued logic.

This is the reason why we are using Belnap's 4-valued logic for the paraconsistent data integration [38] of partially inconsistent and incomplete information. In [38] we use the 4-valued logic directly with Moore's *autoepistemic* operator [25],  $\mu : \mathcal{B} \rightarrow \mathcal{B}$ , for a Belnap's bilattice, defined by  $\mu(x) = t$  if  $x \in \{\top, t\}$ ;  $f$  otherwise.

It is easy to verify that it is monotone w.r.t.  $\leq_t$ , that is, it is multiplicative ( $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ ) and  $\mu(t) = t$  and additive ( $\mu(x \vee y) = \mu(x) \vee \mu(y)$  and  $\mu(f) = f$ ). Consequently, it is a selfadjoint (contemporary universal and existential) modal operator,  $\mu = \neg_t \mu \neg_t$ . But if we are adopting, alternatively, the proposed 2-valued reduction for this Belnap's 4-valued logic, we are able to use the modal operators  $[\perp]$  and  $[\top]$  in order to deal with incomplete and inconsistent information as well.

## 4.2 Binary modal operators case

In this subsection we will use an alternative method w.r.t. the precedent case, based on a flattening, in order to reduce a many-valued into a 2-valued logic. The flattening of an original many-valued lattice-based program into a modal meta logic is a kind of *ontological-encapsulation*, where the encapsulation of an original many-valued logic program into the 2-valued modal meta logic program corresponds to a flattening process described in Definition 5. This approach is developed in a number of papers, and more information can be found in [2,10,39,17]. Here we will present a slightly modified version of this ontological encapsulation.

We will also introduce a new symbol  $\mathbf{e}$  (for "error condition"), necessary in order to render *complete* the functions for a generalized interpretation and a semantic-reflection, defined w.r.t. a particular model  $I_{mv} \in Mod$ , as follows:

**Definition 9.** Let  $PR$  be a many-valued logic program with a set of predicate and functional symbols  $P$  and  $F$  respectively, with a Herbrand model  $I_{mv} : H \rightarrow \mathcal{W}$  where  $H$  is a Herbrand base, with a set  $\mathcal{T}_0$  of all ground terms and a set  $\mathcal{T} = \bigcup_{k \in \mathcal{N}} \mathcal{T}_0^k$  with  $\mathcal{N} = \{1, 2, \dots, n\}$  where  $n$  is the maximal arity of symbols in  $P \cup F$ .

A generalized interpretation is a mapping  $\mathcal{I} : P \times \mathcal{T} \rightarrow \mathcal{W} \cup \{\mathbf{e}\}$ , such that for any  $\mathbf{c} = (c_1, \dots, c_n) \in \mathcal{T}$ ,  $\mathcal{I}(p, \mathbf{c}) = I_{mv}(p(\mathbf{c}))$  if  $ar(p) = n$ ;  $\mathbf{e}$  otherwise.

Then, a semantic-reflection is defined by a mapping  $\mathcal{K} = \lambda \mathcal{I} : P \rightarrow (\mathcal{W} \cup \{\mathbf{e}\})^{\mathcal{T}}$ , where  $\lambda$  is the currying operator from lambda calculus.

For each  $p \in P$  that is not a built-in 2-valued predicate, we define a new functional symbol  $\kappa_p$  for a mapping  $\mathcal{K}(p) : \mathcal{T} \rightarrow \mathcal{W} \cup \{\mathbf{e}\}$ .

If  $p$  is a 2-valued built-in predicate, then the mapping  $\kappa_p$  is defined uniquely and independently of  $I_{mv}$ , by: for any  $\mathbf{c} \in \mathcal{T}_0^{ar(p)}$ ,  $\kappa_p(\mathbf{c}) = 1$  if  $p(\mathbf{c})$  is true; 0 otherwise.

We recall the well-known fact that 2-valued *built-in* predicates (as  $\leq$ ,  $=$ , etc..) have constant extensions in any Herbrand interpretation (they preserve *the same meaning* for any logic interpretation, differently from ordinary predicates).

A semantic-reflection  $\mathcal{K}$ , obtained from a generalized interpretation  $\mathcal{I}$ , introduces a function symbol  $\kappa_p = \mathcal{K}(p)$  for each predicate  $p \in P$  of the original logic program  $PR$ , such that for any  $\mathbf{c} = (c_1, \dots, c_n) \in \mathcal{T}$ , it holds that  $\kappa_p(\mathbf{c}) = I_{mv}(p(\mathbf{c}))$  if  $ar(p) = n$ ;  $\{\mathbf{e}\}$  otherwise. These new function symbols will be used in a new meta logic language, used to transform each original many-valued atom  $p$  in  $P$  into a new atom  $p_F$  obtained as an extension of the original atom  $p$  by one "logic" attribute with the domain of values in  $\mathcal{W}$ . The interpretation of a function symbol  $\kappa_p$  in this new meta logic program has to reflect the meaning of the original many-valued predicate  $p$  in the original many-valued logic program  $PR$ . This is the main reason why we are using the name *semantic-reflection* for a mapping  $\mathcal{K}$ , because by introducing the many-valued interpretations contained in the set of built-in functional symbols  $\kappa_p$  as objects of a meta logic (defined in the following Definition 10), the obtained logic becomes a *meta-logic* w.r.t. the original many-valued logic. Consequently, we are able to introduce a program encapsulation (flattening) transformation  $\mathcal{E}$ , similarly as in [2], as follows:

**Definition 10.** (*Ontological encapsulation of Many-valued Logic Programs: Syntax*)

Let  $PR$  be a many-valued logic program with a set of predicate symbols  $P$ , a many-valued Herbrand model  $I_{mv} : H \rightarrow \mathcal{W}$ , and a semantic-reflection  $\mathcal{K}$ . Then, the translation  $\mathcal{E}$  of a program  $PR$  into its encapsulated syntax version  $PR_F$  is as follows: for each predicate symbol  $p \in P$  with arity  $n$ , we introduce a predicate symbol  $p_F$  with one more attribute with a domain in  $\mathcal{W}$ . Then,

1. each atom  $p(t_1, \dots, t_n)$  in  $PR$  with terms  $t_1, \dots, t_n$ , we transform as follows

$$\mathcal{E}(p(t_1, \dots, t_n)) = p_F(t_1, \dots, t_n, \kappa_p(t_1, \dots, t_n)),$$

and we denote by  $P_F$  the set of all new obtained predicates  $p_F$ .

For any formula  $\phi, \varphi \in \mathcal{L}_{mv}$  we do as follows:

$$2. \mathcal{E}(\sim \phi) = \sim^A \mathcal{E}(\phi);$$

$$3. \mathcal{E}(\phi \wedge \varphi) = \mathcal{E}(\phi) \wedge^A \mathcal{E}(\varphi); \quad \mathcal{E}(\phi \vee \varphi) = \mathcal{E}(\phi) \vee^A \mathcal{E}(\varphi); \quad \mathcal{E}(\phi \leftarrow \varphi) = \mathcal{E}(\phi) \leftarrow^A \mathcal{E}(\varphi),$$

where  $\wedge^A, \vee^A$  and  $\leftarrow^A$  are new introduced binary symbols for the conjunction, disjunction and implication, at the encapsulated meta level, respectively. Thus, the obtained meta program  $PR_F = \{\mathcal{E}(\phi) \mid \phi \text{ is a clause in } PR\}$ , has a Herbrand base  $H_F = \{p_F(c_1, \dots, c_n, \alpha) \mid p(c_1, \dots, c_n) \in H \text{ and } \alpha \in \mathcal{W}\}$ .

We denote by  $\mathcal{L}_F$  the set of formulae (free algebra) obtained from the set of predicate letters in  $P_F$  and modal operators  $\sim^A, \wedge^A, \vee^A$  and  $\leftarrow^A$ .

**Remark:** the new introduced logic symbols  $\sim^A, \wedge^A, \vee^A$  and  $\leftarrow^A$  for the metalogic operators of negation, conjunction, disjunction and implication are not necessarily truth-functional as are original many-valued operators ( $\wedge$  and  $\leftarrow$  for example) but rather are modal (non truth-functional). The unary operator  $\sim^A$  is not a negation (antitonic) operator but a modal operator, so that by this transformation of  $PR$  we obtain a modal logic program  $RP_F$  that is a *positive* logic program (without negation). Differently from a ground many-valued formula  $\phi \in \mathcal{L}_{mv}$ , the transformed meta-formula  $\mathcal{E}(\phi) \in \mathcal{L}_F$  can be only *true* or *false* in a given possible world  $w \in \mathcal{W}$  for this meta modal logic (in a given Kripke model  $\mathcal{M}$  of obtained meta logic program  $PR_F$ ).

In this definition of a meta logic program  $PR_F$ , the set of mappings  $\{\kappa_p = \mathcal{K}(p) \mid p \in$

$P_S\}$  is considered as a set of *built-in functions*, determined by a given semantic reflection  $\mathcal{K}$ , that extends a given set of functional symbols in  $F$ .

This embedding of a many-valued logic program  $P$  into a meta logic program  $P_F$  is an *ontological* embedding: it considers both the formulae of  $PR$  with their many-valued interpretation obtained by semantic reflection (a set of built-in functions  $\kappa_p$ ) of original many-valued logic in this new modal meta logic.

The encapsulation operator  $\mathcal{E}$  is intended to have the following property for a valuation  $v$  (a homomorphic extension of Herbrand interpretation  $I_{mv}$  to all formulae in  $\mathbf{L}_{mv}$  given by Definition 6) of a many-valued logic program  $PR$ :

for any ground many-valued formula  $\phi$ , the encapsulated meta formula  $\mathcal{E}(\phi)$  intends to capture the notion of  $\phi$  with its value  $v(\phi)$  as well, in the way that " $\mathcal{E}(\phi)$  is true exactly in the possible world  $w = v(\phi)$ ".

In order to introduce a concept of *absolute truth or falsity* (not relative to a single possible world in  $\mathcal{W}$ ) for the ground meta formulae in  $\mathbf{L}_F$ , we need a kind of autoepistemic modal operator  $\Diamond$  (it is not part of a language  $\mathbf{L}_F$  obtained by ontological encapsulation). Consequently, for any given ground formula  $\Phi \in \mathbf{L}_F$ , similarly to Moore's autoepistemic operator, a formula  $\Diamond\Phi$  is able to capture the 2-valued notion of " $\Phi$  is a semantic reflection of a many-valued logic program model  $I_{mv}$ ".

Notice that in this encapsulation, for example, the meta-implication  $\leftarrow^A$  derived from the many-valued implication,  $\mathcal{E}(\phi) \leftarrow^A \mathcal{E}(\psi) = \mathcal{E}(\phi \leftarrow \psi)$ , specifies how, for a given clause in  $PR$ , a logic value of the body "propagates" to the head of this clause. It is not functionally dependent on the truth values of its arguments, thus it must be a binary *modal* operator. A Kripke semantics for this *binary* modal operators can be defined based on the simple idea of transforming the many-valued lattice-based operator  $\rightarrow$  into the ternary accessibility relations  $\mathcal{R}_{\rightarrow}$ . The idea to use ternary relations to model binary modal operators comes from Relevance logic [40,41,42,43], but, as far as we know, this is the first time that ternary relations have been built directly from the truth-tables for multi-valued binary logic operators.

**Definition 11.** Let  $PR$  be a many-valued logic program with a set of predicate symbols  $P$ , a many-valued Herbrand model  $I_{mv} : H \rightarrow \mathcal{W}$  and its semantic-reflection  $\mathcal{K}$ .

Then, the model of the flattened program  $PR_F$  in Definition 10 is defined as the Kripke-style model  $\mathcal{M} = (\mathcal{W}, \{\mathcal{R}_{\sim}, \mathcal{R}_{\wedge}, \mathcal{R}_{\vee}, \mathcal{R}_{\rightarrow}, \mathcal{R}_{\times} = \mathcal{W} \times \mathcal{W}\}, S, V)$ , where,

$\mathcal{R}_{\wedge} = \{(x \wedge y, x, y) \mid x, y \in \mathcal{W}\}$ ,  $\mathcal{R}_{\vee} = \{(x \vee y, x, y) \mid x, y \in \mathcal{W}\}$ ,

$\mathcal{R}_{\rightarrow} = \{(x \rightarrow y, x, y) \mid x, y \in \mathcal{W} \text{ and } x \leq y\}$ ,  $\mathcal{R}_{\sim} = \{(\sim x, x) \mid x \in \mathcal{W}\}$ ,

and  $V : \mathcal{W} \times P_F \rightarrow \bigcup_{n \in \mathcal{N}} 2^{S^n \times \mathcal{W}}$  (from Definition 1), such that for any  $p \in P$  with arity  $n$  (i.e.,  $p_F \in P_F$  with arity  $n + 1$ ), a tuple of constants  $(c_1, \dots, c_n) \in S^n$ , and a world  $w \in \mathcal{W}$ ,  $V(w, p_F)(c_1, \dots, c_n, \alpha) = 1$  iff  $w = \alpha = \kappa_p(c_1, \dots, c_n)$ ,

such that, for any formula  $\Phi, \Psi \in \mathbf{L}_F$ , the satisfaction relation  $\models_{w,g}$ , for a given assignment  $g$  and a world  $w \in \mathcal{W}$ , is defined as follows:

1.  $\mathcal{M} \models_{w,g} p_F(x_1, \dots, x_n, \alpha)$  iff  $V(w, p_F)(g(x_1), \dots, g(x_n), \alpha) = 1$ .
2.  $\mathcal{M} \models_{w,g} \sim^A \Phi$  iff  $\exists y((w, y) \in \mathcal{R}_{\sim} \text{ and } \mathcal{M} \models_{y,g} \Phi)$ .
3.  $\mathcal{M} \models_{w,g} \wedge^A(\Phi, \Psi)$  iff  $\exists y, z((w, y, z) \in \mathcal{R}_{\wedge} \text{ and } \mathcal{M} \models_{z,g} \Phi \text{ and } \mathcal{M} \models_{y,g} \Psi)$ .
4.  $\mathcal{M} \models_{w,g} \vee^A(\Phi, \Psi)$  iff  $\exists y, z((w, y, z) \in \mathcal{R}_{\vee} \text{ and } \mathcal{M} \models_{z,g} \Phi \text{ and } \mathcal{M} \models_{y,g} \Psi)$ .
5.  $\mathcal{M} \models_{w,g} \leftarrow^A(\Phi, \Psi)$  iff  $\exists y, z((w, y, z) \in \mathcal{R}_{\rightarrow} \text{ and } \mathcal{M} \models_{z,g} \Phi \text{ and } \mathcal{M} \models_{y,g} \Psi)$ .
6.  $\mathcal{M} \models_{w,g} \Diamond\Phi$  iff  $\exists y((w, y) \in \mathcal{R}_{\times} \text{ and } \mathcal{M} \models_{y,g} \Phi)$ .

The binary operators  $\wedge^A, \vee^A$  and  $\leftarrow^A$  for this multi-modal logic are the existential modal operators w.r.t. the ternary relation  $\mathcal{R}_\wedge, \mathcal{R}_\vee$  and  $\mathcal{R}_\rightarrow$  respectively, while  $\sim^A$  and  $\Diamond$  are the existential unary modal operator w.r.t the binary relation  $\mathcal{R}_\sim$  and  $\mathcal{R}_\times$ , respectively.

Instead of  $\wedge^A(\mathcal{E}(\phi), \mathcal{E}(\psi)), \vee^A(\mathcal{E}(\phi), \mathcal{E}(\psi))$  and  $\leftarrow^A(\mathcal{E}(\phi), \mathcal{E}(\psi))$ , we will use also  $\mathcal{E}(\phi) \wedge^A \mathcal{E}(\psi), \mathcal{E}(\phi) \vee^A \mathcal{E}(\psi)$  and  $\mathcal{E}(\phi) \leftarrow^A \mathcal{E}(\psi)$  respectively.

**Proposition 4** *For any assignment  $g$  and a formula  $\Phi \in \mathbb{L}_F$  we have that  $|\Diamond\Phi/g| \in \{\emptyset, \mathcal{W}\}$ , where  $\emptyset$  is the empty set. That is, for any many-valued formula  $\phi \in \mathbb{L}$  the formula  $\Diamond\mathcal{E}(\phi/g)$  is true in the Kripke-style relational model  $\mathcal{M}$  given by Definition 11, so that  $\mathcal{M}$  is a Kripke-style model of  $PR_F$  correspondent to the many-valued algebraic model  $I_{mv}$  of the original program  $PR$ .*

**Proof:** In what follows we denote by  $v$  the (homomorphic) extension of a Herbrand model  $I_{mv}$  to all ground formulae in  $I_{mv}$ , as defined in Definition 6.

Let us demonstrate that for any  $\phi \in \mathbb{L}$ , i.e.,  $\mathcal{E}(\phi) \in \mathbb{L}_F$ , holds that

$\mathcal{M} \models_{w,g} \mathcal{E}(\phi)$  iff  $w = v(\phi/g)$ .

1. For any atomic formula  $p(x_1, \dots, x_n)$  we have that,

$\mathcal{M} \models_{w,g} \mathcal{E}(p(x_1, \dots, x_n))$  iff  $V(w, p_F)(g(x_1), \dots, g(x_n), \kappa_p(g(x_1), \dots, g(x_n))) = 1$   
iff  $w = \kappa_p(g(x_1), \dots, g(x_n)) = \lambda \mathcal{I}(p)(g(x_1), \dots, g(x_n)) = I_{mv}(p(g(x_1), \dots, g(x_n))) = v(p(x_1, \dots, x_n)/g)$ . Viceversa, if  $w = v(p(x_1, \dots, x_n)/g)$ , i.e.,  $w = I_{mv}(p(x_1, \dots, x_n)/g) = \kappa_p(g(x_1), \dots, g(x_n))$ , then  $V(w, p_F)(g(x_1), \dots, g(x_n), \kappa_p(g(x_1), \dots, g(x_n))) = 1$  and, consequently, from point 1 of definition above,  $\mathcal{M} \models_{w,g} \mathcal{E}(p(x_1, \dots, x_n))$ .

Suppose, by the inductive hypothesis, that  $\mathcal{M} \models_{z,g} \mathcal{E}(\phi)$  iff  $z = v(\phi/g)$ , and

$\mathcal{M} \models_{y,g} \mathcal{E}(\psi)$  iff  $y = v(\psi/g)$ , then:

2. For any formula  $\varphi = \sim \phi$ , we have that  $\mathcal{M} \models_{w,g} \mathcal{E}(\varphi)$  iff  $\mathcal{M} \models_{w,g} \mathcal{E}(\sim \phi)$  iff

$\mathcal{M} \models_{w,g} \sim^A \mathcal{E}(\phi)$  iff  $(\exists z((w, z) \in \mathcal{R}_\sim \text{ and } \mathcal{M} \models_{z,g} \mathcal{E}(\phi)))$ , that is, if

$w = \sim z$  (from the definition of accessibility relation  $\mathcal{R}_\sim$ )

$= \sim v(\phi/g) = v(\sim \phi/g)$  (from a homomorphic property of  $v$ )

$= v(\varphi/g)$ .

Viceversa, if  $w = v(\varphi/g) = v(\sim \phi/g) = \sim v(\phi/g) = \sim z$  then, from the inductive hypothesis,  $\mathcal{M} \models_{w,g} \sim^A \mathcal{E}(\phi)$ , i.e.,  $\mathcal{M} \models_{w,g} \mathcal{E}(\varphi)$ .

3. For any formula  $\varphi = \phi \odot \psi$ , where  $\odot \in \{\wedge, \vee, \rightarrow\}$ , we have that  $\mathcal{M} \models_{w,g} \mathcal{E}(\varphi)$

iff  $\mathcal{M} \models_{w,g} \mathcal{E}(\phi \odot \psi)$  iff  $(\mathcal{M} \models_{w,g} \mathcal{E}(\phi) \odot^A \mathcal{E}(\psi) \text{ iff } (\exists y, z((w, y, z) \in \mathcal{R}_\odot \text{ and } \mathcal{M} \models_{z,g} \mathcal{E}(\phi) \text{ and } \mathcal{M} \models_{y,g} \mathcal{E}(\psi))))$ , that is, if

$w = z \odot y$  (from a definition of accessibility relation  $\mathcal{R}_\odot$ )

$= v(\phi/g) \odot v(\psi/g) = v(\phi/g \odot \psi/g)$  (from a homomorphic property of  $v$ )

$= v((\phi \odot \psi)/g) = v(\varphi/g)$ .

Viceversa, if  $w = v(\varphi/g) = v(\phi \odot \psi)/g = v(\phi/g) \odot v(\psi/g) = z \odot y$  then, from the inductive hypothesis,  $\mathcal{M} \models_{w,g} \mathcal{E}(\phi) \odot^A \mathcal{E}(\psi)$ , i.e.,  $\mathcal{M} \models_{w,g} \mathcal{E}(\varphi)$ .

Thus, for any  $\Phi \in \mathbb{L}_F$  we have that  $|\Phi/g| = \{w\}$  for some  $w \in \mathcal{W}$ , if  $\Phi/g = \mathcal{E}(\phi/g)$ ; otherwise  $|\Phi/g| = \emptyset$ .

Consequently, we have that  $|\Diamond\Phi/g| = \{w \mid \exists y((w, y) \in \mathcal{R}_\times \text{ and } \mathcal{M} \models_{y,g} \Phi)\} = \mathcal{W}$  if  $\Phi/g = \mathcal{E}(\phi/g)$ ; otherwise  $|\Diamond\Phi/g| = \emptyset$ . That is, each ground modal formula  $\Diamond\Phi/g$  for any  $\Phi \in \mathbb{L}_F$  is a 2-valued formula.

From Definition 11 we have seen how a many-valued model  $I_{mv}$  of a logic program  $PR$

uniquely determines a Kripke model  $\mathcal{M}$  of its meta-logic modal program  $PR_F$ . Let us now show the opposite direction, that is, how a Kripke model  $\mathcal{M}$  of a modal logic program  $PR_F$  obtained by ontological encapsulation of the original many-valued logic program  $PR$ , determines uniquely a many-valued model  $I_{mv}$  of the logic program  $PR$ . That is, let us show that the set of ground atomic modal formulae  $\Diamond p_F(c_1, \dots, c_n, \alpha)$  for  $p_F(c_1, \dots, c_n, \alpha) \in H_F$ , which are *true* in a Kripke model  $\mathcal{M}$ , uniquely determines the many-valued Herbrand model  $I_{mv}$  of the original logic program  $PR$ :

In fact, we define uniquely the mapping  $I_{mv} : H \rightarrow \mathcal{W}$ , as follows: for any modal atomic formula  $\Diamond p_F(c_1, \dots, c_n, \alpha)$ , *true* in the Kripke model  $\mathcal{M}$ , we define  $I_{mv}(p(c_1, \dots, c_n)) = \alpha$ . It is easy to verify that such a definition of a mapping  $I_{mv} : H \rightarrow \mathcal{W}$  is a Herbrand model of a many-valued logic program  $PR$ .

□

This transformation of multi-valued logic programs into 2-valued multi-modal logic programs can be briefly explained as follows: we transform the original multi-valued atoms into the meta 2-valued atoms by enlarging the original atoms with a new logic attribute with the domain of values in  $\mathcal{W}$ . This ontological encapsulation also eliminates the negation (in this case the negation  $\sim$ ) by introducing a unary modal operator  $\sim^A$ . The remained binary multi-valued lattice operations are substituted by the 2-valued binary modal operators, by transforming the truth functional tables of these operators directly into the ternary accessibility relations of this modal logic.

**Remark:** In addition, this ontological encapsulation of logic programs into the *positive* (without the negation) modal programs, can be used, with some opportune modifications of the definitions above where a ground atom  $p_F(c_1, \dots, c_n, \alpha) \in H_F$  is true only for exactly one value  $\alpha \in \mathcal{W}$ , to deal with the *inconsistency* of 2-valued logic programs: the resulting positive modal program will be a *paraconsistent* logic program, that is for any given ground atom  $p(c_1, \dots, c_n)$  of the original 2-valued logic program that is inconsistent (both true and false), in the transformed *consistent* positive modal program we can (consistently) have two true ground atoms,  $p_F(c_1, \dots, c_n, 1)$  and  $p_F(c_1, \dots, c_n, 0)$ .

The relationship between these two program transformations, for finite and infinite cases of many-valued programs, can be given by the following corollary:

**Corollary 1.** *For any atom  $p(x_1, \dots, x_n)$  of a many-valued logic program and its two 2-valued program transformations defined previously, the following semantic connection holds  $\mathcal{M} \models_{w,g} \mathcal{E}(p(x_1, \dots, x_n))$  iff  $\llbracket w \rrbracket p(g(x_1), \dots, g(x_n))$  is true in  $\mathcal{M}_I$ .*

Consequently, we can conclude that many-valued logic programs can be equivalently replaced by positive 2-valued multi-modal logic programs, and this reduction of many-valued logics into modal logics also explains the good properties of many-valued logic programs.

Moreover, we have shown that by a 2-valued reduction of many-valued *logic programs* we obtain a 2-valued *non-truth-functional* logic, and that such a logic is just a 2-valued (multi)modal logic with a non-standard autoreferential Kripke semantics, because modal operators are generally *non monotonic*, and in a second case we need also *binary* modal operators.

In what follows we will generalize this 2-valued reduction presented for only Logic Programs, to any kind of many-valued logics.



## 5 A general abstract reduction of many-valued into 2-valued logics

The term "abstract" used for this general many-valued reduction means that we do not consider any further the specific reduction of particular functional logic operators in  $\Sigma$  of a many-valued logic into correspondent modal operators, but rather a general reduction independent of them, based on structural consequence operations or matrices.

As we will see, both abstract reductions will result in a kind of 2-valued modal logic that are not truth-functional, as we obtained in the specific case for Logic Programs in Section 4.

In [44] Suszko's thesis was presented. This paper is extremely dense and very short, and thus it is not easy to understand; it is a kind of synthesis, in four pages, of some deep reflections carried out by Suszko over forty years. Only 15 years after this publication, Malinowski's book [45] has thrown some light on it (see especially Chapter 10, Section 10.1). Unfortunately, neither the quoted paper by Suszko nor Malinowski's book explicitly state Suszko's thesis, but in another paper [46] Malinowski has written "Suszko's thesis ... states that each logic, i.e., a *structural consequence operation* conforming Tarski's conditions, is logically two-valued", and (p.73) "each (structural) *propositional logic*  $(L, C)$  can be determined by a class of logical valuations of the language  $L$  or, in other words, it is logically *two-valued*".

In what follows we will try to formally develop a reduction of a many-valued *predicate* logic  $L_{mv}$ , with a Herbrand base  $H$ , into a 2-valued logic, based on these observations of Suszko.

We denote, for a given set of thesis (ground formulae)  $\Gamma$  of a many-valued logic  $L_{mv}$ , the 2-valued structural consequence relation by  $\Gamma \vdash \phi$ , which means that a ground formula  $\phi$  is a structural consequence of set of ground formulae in  $\Gamma$ , i.e., that  $\phi \in C(\Gamma)$  where  $C$  is a structural consequence operation conforming Tarski's conditions.

We denote by  $Val = \mathcal{B}^H$  the set of Herbrand many-valued interpretations  $v : H \rightarrow \mathcal{B}$ ,  $v \in Val$ , for a many-valued logic  $L_{mv}$  with a Herbrand base  $H$  and a set of *algebraic* truth-values in  $\mathcal{B}$ . Let  $Val_\Gamma \subset Val$  be a non-empty subset of *models* of  $\Gamma$ , that is, valuations  $v \in Val_\Gamma$  that satisfy every ground formula in  $\Gamma$ .

Then, the truth of  $\Gamma \vdash \phi$  is equivalent to the fact that every valuation  $v \in Val_\Gamma$  is a model of  $\phi$  also (i.e., satisfies a ground formula  $\phi$ ). However, here we are not speaking about a truth value of a many-valued ground formulae  $\phi \in L_{mv}$ , but about a truth value of a meta sentence  $\Gamma \vdash \phi$ . In what follows, for a fixed set of (initial) thesis  $\Gamma \subset L_{mv}$  that defines a structural many-valued logic  $(\Gamma, C)$ , we will transform the left side construct  $\Gamma \vdash (\_)$  in an universal *modal operator*  $\Box_\Gamma$  (" $\Gamma$ -deducible"), so that a meta sentence  $\Gamma \vdash \phi$  can be replaced by an equivalent modal formula  $\Box_\Gamma \phi$  in this 2-valued meta logic.

Thus, analogously to the more specific cases for Logic Programs, also in this general abstract 2-valued reduction we are not speaking about the two-valuedness of an original *many-valued formula*, but about a *modal formula* of a 2-valued meta-logic obtained by this transformation.

What remains now is to define a Kripke semantics for this modal meta-logic, denoted by  $L_{\mathcal{F}}$ , obtained from a set of formulae  $\mathcal{F} = \{\Box_\Gamma \phi \mid \phi \in L_{mv}\}$  and the standard 2-valued logic connectives (conjunction, disjunction, implication and negation).

**Definition 12.** Given a structural many-valued logic  $(\Gamma, C)$ , where  $\Gamma \subset \mathbf{L}_{mv}$  is a subset of ground formulae with a set of predicate symbols in  $P$  and a Herbrand base  $H$ , we define a Kripke-style model for Suszko's reduction,  $\mathcal{M} = (\mathcal{W}, \mathcal{R}_\Gamma, S, V)$ , where a set of possible worlds is  $\mathcal{W} = Val$ ,  $\mathcal{R}_\Gamma = Val \times Val_\Gamma$ , and  $V : \mathcal{W} \times P \rightarrow \bigcup_{n \in \mathcal{N}} 2^{S^n \times \mathcal{W}}$  (from Definition 1), such that for any  $p \in P$  with arity  $n$ , a tuple of constants  $(c_1, \dots, c_n) \in S^n$ , and a world  $w \in \mathcal{W}$ , (a Herbrand interpretation  $w : H \rightarrow \mathcal{B}$ ),  $V(w, p)(c_1, \dots, c_n) = 1$  iff  $w \in Val_\Gamma$ .

The satisfaction relation  $\models_{w,g}$ , for a given assignment  $g$  and a world  $w \in \mathcal{W}$ , for any many-valued formula  $\phi, \psi$ , is defined as follows:

1.  $\mathcal{M} \models_{w,g} p(x_1, \dots, x_n)$  iff  $V(w, p)(g(x_1), \dots, g(x_n)) = 1$ .
  2.  $\mathcal{M} \models_{w,g} \phi$  iff the homomorphic extension (in Definition 6) of the Herbrand model  $w$  is a model of the ground formula  $\phi/g$ .
  3.  $\mathcal{M} \models_{w,g} \Box_\Gamma \phi$  iff  $\forall w' ((w, w') \in \mathcal{R}_\Gamma \text{ implies } \mathcal{M} \models_{w',g} \phi)$ .
  4.  $\mathcal{M} \models_{w,g} \neg \Box_\Gamma \phi$  iff not  $\mathcal{M} \models_{w,g} \Box_\Gamma \phi$ ,
  5.  $\mathcal{M} \models_{w,g} \Box_\Gamma \phi \wedge \Box_\Gamma \psi$  iff  $\mathcal{M} \models_{w,g} \Box_\Gamma \phi$  and  $\mathcal{M} \models_{w,g} \Box_\Gamma \psi$ ,
  6.  $\mathcal{M} \models_{w,g} \Box_\Gamma \phi \vee \Box_\Gamma \psi$  iff  $\mathcal{M} \models_{w,g} \Box_\Gamma \phi$  or  $\mathcal{M} \models_{w,g} \Box_\Gamma \psi$ ,
  7.  $\mathcal{M} \models_{w,g} \Box_\Gamma \phi \rightarrow \Box_\Gamma \psi$  iff  $\mathcal{M} \models_{w,g} \Box_\Gamma \phi$  implies  $\mathcal{M} \models_{w,g} \Box_\Gamma \psi$ ,
- where the logic connectives  $\wedge, \vee, \rightarrow$  and  $\neg$  are the classic 2-valued conjunction, disjunction, implication and negation respectively.

Notice that a satisfaction of the 2-valued formulae of this meta-logic  $\mathbf{L}_{\mathcal{F}}$ , obtained by Suszko's reduction of the original many-valued logic, is relative to points 3 to 7 in the Definition above. Consequently, the two-valuedness is a property not of the original many-valued formulae, but of the modal formulae in this non truth-functional modal meta-logic. Let us show that this reduction is sound and complete.

**Lemma 1.** Given a Kripke model  $\mathcal{M} = (\mathcal{W}, \mathcal{R}_\Gamma, S, V)$  in Definition 12, for a given many-valued logic  $(\Gamma, C)$ , where  $\Gamma \subset \mathbf{L}_{mv}$  is a subset of ground formulae, then for any formula  $\phi \in \mathbf{L}_{mv}$  and assignment  $g$  we have that:  
 $\phi/g \in C(\Gamma)$ , (i.e.,  $\Gamma \vdash \phi/g$ ) iff  $\Box_\Gamma \phi/g$  is true in  $\mathcal{M}$ .

**Proof:** If  $\Gamma \vdash \phi/g$  then for every  $w \in Val_\Gamma$  its homomorphic extension to all ground formulae in  $\mathbf{L}_{mv}$  is a model of a ground formula  $\phi/g \in \mathbf{L}_{mv}$ . Thus,

$$\begin{aligned} |\Box_\Gamma \phi/g| &= \{w \mid \forall w' ((w, w') \in \mathcal{R}_\Gamma \text{ implies } \mathcal{M} \models_{w',g} \phi) \} \\ &= \{w \mid \forall w' ((w, w') \in \mathcal{R}_\Gamma \text{ implies } w' \text{ is a model of } \phi/g) \} \\ &= \{w \mid \forall w' (w' \in Val_\Gamma \text{ implies } w' \text{ is a model of } \phi/g) \} \\ &= \{w \mid \text{true} \} = \mathcal{W}, \quad \text{i.e., } \Box_\Gamma \phi/g \text{ is true in } \mathcal{M}. \end{aligned}$$

Viceversa, if  $\Box_\Gamma \phi/g$  is true in  $\mathcal{M}$  then  $\mathcal{W} = |\Box_\Gamma \phi/g| = \{w \mid \forall w' ((w, w') \in \mathcal{R}_\Gamma \text{ implies } \mathcal{M} \models_{w',g} \phi) \} = \{w \mid \forall w' \in Val_\Gamma (w' \text{ is a model of } \phi/g) \}$ , that is, the following sentence has to be true:  $\forall w' \in Val_\Gamma (w' \text{ is a model of } \phi/g)$ , and, consequently,  $\Gamma \vdash \phi/g$ , i.e.,  $\phi/g \in C(\Gamma)$ .

□

These results confirm da Costa's idea [47] that a reduction to 2-valuedness can be done at an abstract level, without taking into account the underlying structure of the set of many-valued formulae (differently from the particular case of Logic Programs given in Section 4).

It is not necessary to make a detour by matrices in order to get this reduction. But in the

case where we have a many-valued logic with a given matrix  $(\mathcal{B}, D)$ , where  $D \subset \mathcal{B}$  is a subset of designated algebraic truth values, then we are able to define a new modal 2-valued reduction for such a many-valued logic, based on the *existential* modal operator  $\Diamond_D$  ("D-satisfied"). It is given in the way that, for given homomorphic extension of a valuation  $v : H \rightarrow \mathcal{B}$ , a many-valued formula  $\phi \in \mathcal{L}_{mv}$  and an assignment  $g$ , the formula  $\Diamond_D \phi / g$  is true iff  $v(\phi / g) \in D$ , that is, iff  $v$  satisfies (is a model of)  $\phi / g$ . What remains now is to define a Kripke semantics for this matrix-based reduction to a modal meta-logic, denoted by  $\mathcal{L}_{\mathcal{E}}$ , obtained from a set of formulae  $\mathcal{E} = \{\Diamond_D \phi \mid \phi \in \mathcal{L}_{mv}\}$  and standard 2-valued logic connectives (conjunction, disjunction, implication and negation).

**Definition 13.** Given a many-valued logic  $\mathcal{L}_{mv}$  with a given matrix  $(\mathcal{B}, D)$ , a set of predicate symbols in  $P$  and a Herbrand base  $H$ , we define a Kripke-style model for a matrix-based reduction by a quadruple  $\mathcal{M} = (\mathcal{W}, \mathcal{R}_D, S, V)$ , where a set of possible worlds is  $\mathcal{W} = \mathcal{B}$ ,  $\mathcal{R}_D = \mathcal{B} \times D$ , and  $V : \mathcal{W} \times P \rightarrow \bigcup_{n \in \mathcal{N}} 2^{S^n \times \mathcal{W}}$  (from Definition 1), such that for any  $p \in P$  with arity  $n$ , a tuple of constants  $(c_1, \dots, c_n) \in S^n$ ,  $V(w, p)(c_1, \dots, c_n) = 1$  for exactly one world  $w \in D \subseteq \mathcal{W}$ .

The satisfaction relation  $\models_{w,g}$ , for a given assignment  $g$  and a world  $w \in \mathcal{W}$ , for any many-valued formula  $\phi, \psi \in \mathcal{L}_{mv}$ , is defined as follows:

1.  $\mathcal{M} \models_{w,g} p(x_1, \dots, x_n)$  iff  $V(w, p)(g(x_1), \dots, g(x_n)) = 1$ .
  2.  $\mathcal{M} \models_{w,g} \phi$  iff  $w = v(\phi / g) \in D$ , where  $v$  is the unique homomorphic extension (Definition 6) of a mapping  $v : H \rightarrow \mathcal{B}$  defined by: for each  $p(c_1, \dots, c_n) \in H$ ,  $v(p(c_1, \dots, c_n)) = y$  such that  $V(y, p)(c_1, \dots, c_n) = 1$ .
  3.  $\mathcal{M} \models_{w,g} \Diamond_D \phi$  iff  $\exists w' ((w, w') \in \mathcal{R}_D \text{ and } \mathcal{M} \models_{w',g} \phi)$ .
  4.  $\mathcal{M} \models_{w,g} \neg \Diamond_D \phi$  iff not  $\mathcal{M} \models_{w,g} \Diamond_D \phi$ ,
  5.  $\mathcal{M} \models_{w,g} \Diamond_D \phi \wedge \Diamond_D \psi$  iff  $\mathcal{M} \models_{w,g} \Diamond_D \phi$  and  $\mathcal{M} \models_{w,g} \Diamond_D \psi$ ,
  6.  $\mathcal{M} \models_{w,g} \Diamond_D \phi \vee \Diamond_D \psi$  iff  $\mathcal{M} \models_{w,g} \Diamond_D \phi$  or  $\mathcal{M} \models_{w,g} \Diamond_D \psi$ ,
  7.  $\mathcal{M} \models_{w,g} \Diamond_D \phi \rightarrow \Diamond_D \psi$  iff  $\mathcal{M} \models_{w,g} \Diamond_D \phi$  implies  $\mathcal{M} \models_{w,g} \Diamond_D \psi$ ,
- where the logic connectives  $\wedge, \vee, \rightarrow$  and  $\neg$  are the classic 2-valued conjunction, disjunction, implication and negation respectively.

Notice that in this case we obtained an autoreferential semantics [31,32] and that a satisfaction of the 2-valued formulae of this meta-logic  $\mathcal{L}_{\mathcal{E}}$ , obtained by the matrix-based reduction of original many-valued logic, is relative to points 3 to 7 in the Definition above. Consequently, the two-valuedness is a property not of the original many-valued formula, but of the modal formula in this non truth-functional modal meta-logic.

Let us show that this matrix-based reduction is sound and complete.

**Lemma 2.** Let  $\mathcal{M} = (\mathcal{W}, \mathcal{R}_D, S, V)$  be a Kripke model, given in Definition 13, for a many-valued logic  $\mathcal{L}_{mv}$  with a matrix  $(\mathcal{B}, D)$ . We define a many-valued Herbrand interpretation  $v : H \rightarrow \mathcal{B}$  as follows: for each  $p(c_1, \dots, c_n) \in H$ ,  $v(p(c_1, \dots, c_n)) = w$ , where  $w$  is the unique value that satisfies  $V(w, p)(c_1, \dots, c_n) = 1$ . Then, for any formula  $\phi \in \mathcal{L}_{mv}$  and an assignment  $g$ , we have that, "the homomorphic extension of  $v$  is a model of  $\phi / g$ " iff  $\Diamond_D \phi / g$  is true in  $\mathcal{M}$ .

**Proof:** If the homomorphic extension of  $v$  is a model of  $\phi / g$  then  $w' = v(\phi / g) \in D$ , thus,  $|\Diamond_D \phi / g| = \{w \mid \exists w' ((w, w') \in \mathcal{R}_D \text{ and } \mathcal{M} \models_{w',g} \phi)\} = \{w \mid \exists w' ((w, w') \in \mathcal{R}_D \text{ and } w' = v(\phi / g))\} = \{w \mid \text{true}\} = \mathcal{W}$ ,

i.e.,  $\Diamond_D \phi/g$  is true in  $\mathcal{M}$ .

Viceversa, if  $\Diamond_D \phi/g$  is true in  $\mathcal{M}$  then  $\mathcal{W} = |\Diamond_D \phi/g| = \{w \mid \exists w' ((w, w') \in \mathcal{R}_D \text{ and } \mathcal{M} \models_{w',g} \phi)\} = \{w \mid \exists w' \in D (w' = v(\phi/g))\}$ , that is, the following sentence has to be true:  $\exists w' \in D (w' = v(\phi/g))$ , and, consequently, it must hold that  $v(\phi/g) \in D$ , i.e., the homomorphic extension of  $v$  is a model of  $\phi/g$ .

□

## 6 Conclusion

As we mentioned, real-world problems often have to be resolved by applying Artificial Intelligence techniques by means of many-valued logics (fuzzy, paraconsistent, bilattice-based, etc.), therefore, the investigation of the general properties of these non standard many-valued logics is a very important issue. Based on Suszko's thesis, in this paper we analyzed a different possibility of reducing these many-valued logics into 2-valued logics, in order to be able to compare their original many-valued properties based on such obtained 2-valued logic. Our approach, however, is formal and constructive, in contrast to Suszko's nonconstructive approach based on a distinction between designated and undesignated algebraic truth-values.

We introduced a kind of a contextualization for many-valued logics that is similar to the special annotated logics case, but which gives us the possibility of continuing to use the standard Herbrand models as well. In this paper we have shown how many-valued logic programs can be equivalently transformed into contextual logic programs with higher-order Herbrand interpretations. We have shown that the flattening of such higher-order Herbrand interpretations leads to 2-valued logic programs, identical to meta logic programs obtained by an ontological encapsulation of the original Many-valued logic programs [2,10] with modal logic connectives. From the other side, the properties of higher-order Herbrand types, with a possibility of introducing the Kripke semantics for them, are the basis for an equivalent transformation of many-valued Logic Programs into the 2-valued multi-modal Logic Programs with modal atoms.

We also developed a general abstract 2-valued reduction for any kind of many-valued logics, based on informal Suszko's thesis, and have shown the Kripke semantics for obtained 2-valued modal meta-logics, for both Suszko's (non-matrix) and matrix-based cases.

Consequently, any kind of reduction of a many-valued logic into 2-valued logic results in a non truth-functional modal meta-logic, which obviously is not an original "reference" many-valued logic. This process is explained by the fact that this reduction is based on new sentences about the original many-valued sentences, and that, by avoiding the second order syntax of these meta-sentences, what is required is the introduction of new *modal* operators in this equivalent but 2-valued meta-logic. As presented in the case of Logic Programming and general structural many-valued logics, this is a general approach to 2-valued reductions.

This results consolidate an intuition that the many-valued logics, used for uncertain, approximated and context-dependent information, can be embedded into multi-modal logics with possible world semantics, which are well investigated sublanguages of the standard First-order logic language with very useful properties.

This method can be used for *paraconsistent* logics as well, as shown in an example for the 4-valued Belnap' bilattice, and explains why the paraconsistent logics can be formalized by modal logics as well.

Further investigation: It is well known, by Definition 2 in [48,49], that any 2-valued modal logic can be equivalently transformed into a truth-valued many-valued logic with a complete distributive lattice of its "algebraic functional" logic values (so called complex algebras over powerset of possible worlds), as for example the complex algebra for a (modal) intuitionistic logic is a Heyting algebra over the powerset of possible worlds. Here we demonstrated that, additionally, every truth-functional many-valued logic can be reduced into a non truth functional *modal* (meta) logics. There does remain an open question: are all 2-valued non truth-functional logics *necessarily modal* logics? Consider, for example, the paraconsistent da Costa's  $C_n$  system [50] for which the relational Kripke semantics has not still been defined.

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